

On Zero-Sum Stochastic Differential Games

Erhan Bayraktar^{*†}, Song Yao[‡]

Abstract

We generalize the results of Fleming and Souganidis [18] on zero-sum stochastic differential games to the case when the controls are unbounded. We do this by proving a dynamic programming principle using a covering argument instead of relying on a discrete approximation (which is used along with a comparison principle in [18]). Also, in contrast with [18], we define our pay-off through a doubly reflected backward stochastic differential equation. The value function (in the degenerate case of a single controller) is closely related to the second order doubly reflected BSDEs.

Keywords: Zero-sum stochastic differential games, Elliott-Kalton strategies, dynamic programming principle, stability under pasting, doubly reflected backward stochastic differential equations, viscosity solutions, obstacle problem for fully non-linear PDEs, shifted processes, shifted SDEs, second-order doubly reflected backward stochastic differential equations.

Contents

| | | |
|----------|---|-----------|
| 1 | Introduction | 2 |
| 1.1 | Notation and Preliminaries | 4 |
| 1.2 | Doubly Reflected Backward Stochastic Differential Equations | 6 |
| 2 | Stochastic Differential Games with Square-Integrable Controls | 6 |
| 2.1 | Continuous Dependence Results | 7 |
| 2.2 | Definition of the value functions and the Dynamic Programming Principle | 10 |
| 3 | An Obstacle Problem for Fully non-linear PDEs | 12 |
| 4 | Shifted Processes | 13 |
| 4.1 | Concatenation of Sample Paths | 13 |
| 4.2 | Measurability of Shifted Processes | 14 |
| 4.3 | Integrability of Shifted Processes | 14 |
| 4.4 | Shifted Stochastic Differential Equations | 15 |
| 4.5 | Pasting of Controls and Strategies | 16 |
| 5 | Optimization Problems with Square-Integrable Controls | 16 |
| 5.1 | General Results | 17 |
| 5.2 | Connection to the <i>Second-Order</i> Doubly Reflected BSDEs | 17 |
| 6 | Proofs | 20 |
| 6.1 | Proofs of Section 1 & 2 | 20 |
| 6.2 | Proof of Dynamic Programming Principle | 30 |
| 6.3 | Proofs of Section 3 | 37 |

^{*}Department of Mathematics, University of Michigan, Ann Arbor, MI 48109; email: erhan@umich.edu.

[†]E. Bayraktar is supported in part by the National Science Foundation under applied mathematics research grants and a Career grant, DMS-0906257, DMS-1118673, and DMS-0955463, respectively, and in part by the Susan M. Smith Professorship.

[‡]Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260; email: songyao@pitt.edu.

| | | |
|-----|-------------------------------|----|
| 6.4 | Proofs of Section 4 | 45 |
| 6.5 | Proofs of Section 5 | 57 |

1 Introduction

In this paper we use doubly reflected backward stochastic differential equations (DRBSDEs) to generate payoffs for a zero-sum stochastic differential game introduced by the seminal work of Fleming and Souganidis [18]. In our setting, the two players compete by choosing square-integrable controls.

DRBSDEs were first analyzed by Cvitanic and Karatzas [13], who showed that the solution of a DRBSDE is the value of a certain Dynkin game, a zero-sum stochastic game of optimal stopping. Then Hamadène et al. [20, 21, 23, 22, 16] added controls into DRBSDEs to study mixed control and stopping games and saddle point problems, only when the drift is controlled. Recent advances for Dynkin games and controller and stopper games were made by Karatzas et al. [28, 29, 3], Bayraktar et al. [6, 7, 2, 4], by [36]. On the other hand, when there are two competing controllers who can also control the diffusion coefficient, there are a lot of technicalities involved as it is demonstrated by [18]. In particular, *Elliott-Kalton strategies* needs to be used for the controller with lower priority. Recently Buckdahn and Li [9, 11, 10] and Hamadène et al. [25] made some significant advances to these types of games. However, as in [18], they assumed that the control spaces are compact. Also, the analysis in these papers is different than [18] and ours in that they work with a uniform canonical space $\Omega = \{\omega \in \mathbb{C}([0, T]; \mathbb{R}^d) : \omega(0) = 0\}$ regardless of the starting time of the game.

One encounters tremendous technical difficulties when the compactness assumption of the control spaces is removed, since the approximation tool of [18] (also see Fleming and Hernández Hernández [17]) is not applicable any longer. There are some exceptions to this rule: Square-integrable controls was considered by Browne [8] for a specific zero-sum investment game between two small investors whose controls are in form of their portfolios. The PDEs in this special case have smooth solutions, therefore the problem can be solved by relying on a verification theorem instead of the dynamic programming principle. In a more general setting, Chapter 6 of Krylov [30] considered square-integrable controls. However, the analysis was done only for cooperative games (i.e. the so called sup sup case). It is also worth mentioning that inspired by the “tug-of-war” (a discrete-time random turn game, see e.g. [37] and [31]), Atar and Budhiraja [1] studied a zero-sum stochastic differential game with $\mathbb{U} = \mathbb{V} = \{x \in \mathbb{R}^n : |x| = 1\} \times [0, \infty)$ played until the state process exits a given domain. The authors showed that the value of such a game is the unique viscosity solution to the inhomogenous infinity Laplace equation. As in Chapter 6 of [30], they depend on an approximating the game with unbounded controls with a sequence of games with bounded controls. They prove a dynamic programming principle for the latter case and prove the equicontinuity of the approximating sequence to conclude that the value function is a viscosity solution to the infinity Laplace equation. Instead of relying on approximation argument, we directly prove a dynamic programming principle for the game with square-integrable controls.

In this paper, the controls of respective players take values in two separable metric spaces \mathbb{U} and \mathbb{V} . We follow the probabilistic setting of [18] and rely on the existence of the regular conditional probability distributions. When the game starts from time $t \in [0, T]$, we consider the canonical space $\Omega^t \triangleq \{\omega \in \mathbb{C}([t, T]; \mathbb{R}^d) : \omega(t) = 0\}$, whose coordinator process B^t is a Brownian motion under the Wiener measure P_0^t . Denote by \mathcal{U}^t (resp. \mathcal{V}^t) the set of all \mathbb{U} -valued (resp. \mathbb{V} -valued) square-integrable process. If player I chooses a $\mu \in \mathcal{U}^t$ and player II selects a $\nu \in \mathcal{V}^t$ as controls, the state process $X^{t,x,\mu,\nu}$ starting from x will evolve according to the following SDE:

$$X_s = x + \int_t^s b(r, X_r, \mu_r, \nu_r) dr + \int_t^s \sigma(r, X_r, \mu_r, \nu_r) dB_r^t, \quad s \in [t, T], \quad (1.1)$$

where the drift b and the diffusion σ are Lipschitz in x and have linear growth in (μ, ν) . The payoff player I will receive from player II is determined by the first component of the solution $(Y^{t,x,\mu,\nu}, Z^{t,x,\mu,\nu}, \underline{K}^{t,x,\mu,\nu}, \overline{K}^{t,x,\mu,\nu})$ to

the following DRBSDE:

$$\begin{cases} Y_s = h(X_s^{t,x,\mu,\nu}) + \int_s^T f(r, X_r^{t,x,\mu,\nu}, Y_r, Z_r, \mu_r, \nu_r) dr + \underline{K}_T - \underline{K}_s - (\overline{K}_T - \overline{K}_s) - \int_s^T Z_r dB_r^t, & s \in [t, T], \\ \underline{l}(s, X_s^{t,x,\mu,\nu}) \leq Y_s \leq \overline{l}(s, X_s^{t,x,\mu,\nu}), & s \in [t, T], \\ \int_t^T (Y_s - \underline{l}(s, X_s^{t,x,\mu,\nu})) d\underline{K}_s = \int_t^T (\overline{l}(s, X_s^{t,x,\mu,\nu}) - Y_s) d\overline{K}_s = 0, \end{cases} \quad (1.2)$$

with two separate obstacle functions $\underline{l} < \overline{l}$ satisfying $\underline{l}(T, \cdot) \leq h(\cdot) \leq \overline{l}(T, \cdot)$. When $\underline{l}, h, \overline{l}, f$ are all $2/q$ -Hölder continuous in x for some $q \in (1, 2]$, $Y^{t,x,\mu,\nu}$ is q -integrable by El Asri et al. [15]. As we see from (1.1) and (1.2) that the controls μ and ν influence the game in two aspects: either affect (1.2) via the state process $X^{t,x,\mu,\nu}$ or appear directly in the generator f of (1.2) as parameters.

We use Elliott-Kalton strategies as in [18]. In this game, one player (e.g. Player I) has priority and chooses a control and its opponent (e.g. Player II) will react by selecting a corresponding strategy. We specify Player II's strategy by a measurable mapping $\beta : [t, T] \times \Omega^t \times \mathbb{U} \rightarrow \mathbb{V}$ if the game starts from time t . This additional specification is to accommodate a particular measurability issue, see Remark 2.2. Under a linear growth condition on the μ -variable, β induces a mapping $\beta(\cdot) : \mathcal{U}^t \rightarrow \mathcal{V}^t$ by $(\beta(\mu))_r(\omega) \triangleq \beta(r, \omega, \mu_r(\omega))$, $\forall \mu \in \mathcal{U}^t, (r, \omega) \in [t, T] \times \Omega^t$, which is exactly an Elliott-Kalton strategy. Then $w_1(t, x) \triangleq \inf_{\beta \in \mathfrak{B}^t} \sup_{\mu \in \mathcal{U}^t} Y_t^{t,x,\mu,\beta(\mu)}$ represents Player I's priority value of the game starting from time t and state x , where \mathfrak{B}^t collects all admissible strategies for Player II. Player II's priority value $w_2(t, x)$ is defined similarly.

Although value functions $w_1(t, x), w_2(t, x)$ are still $2/q$ -Hölder continuous in x , they are no longer $1/q$ -Hölder continuous in t as in the case of compact control spaces. Hence we are not able to use the approach of [18] to show the dynamic programming principle for w_1 and w_2 ; see Remark 2.2. Instead, we use the continuity of $Y^{t,x,\mu,\nu}$ in controls (μ, ν) , properties of shifted processes (especially shifted SDEs) as well as stability under pasting of controls/strategies (as listed below) to prove a dynamic programming principle, say for w_1 :

$$w_1(t, x) = \inf_{\beta \in \mathfrak{B}^t} \sup_{\mu \in \mathcal{U}^t} Y_t^{t,x,\mu,\beta(\mu)} \left(\tau_{\mu,\beta}, w_1(\tau_{\mu,\beta}, X_{\tau_{\mu,\beta}}^{t,x,\mu,\beta(\mu)}) \right) \quad (1.3)$$

for any family $\{\tau_{\mu,\beta} : \mu \in \mathcal{U}^t, \beta \in \mathfrak{B}^t\}$ of \mathbb{Q} -valued stopping times. The crucial ingredients of the proof of dynamic programming principle (1.3) are:

i) When b, σ are λ -Hölder continuous in μ (or ν) and f is 2λ -Hölder continuous in μ (or ν) for some $\lambda \in (0, 1)$, applying an a priori estimate (1.7) for DRBSDEs, we obtain a continuous dependence result for $X^{t,x,\mu,\nu}$ and $Y^{t,x,\mu,\nu}$ on control μ (or ν), see Lemma 2.1 and Lemma 2.2. This dependence together with two nice topological properties of the canonical spaces Ω^t , namely separability and Lemma 6.3 are crucial in the covering argument which is used to construct ε optimal strategies starting at any stopping time.

ii) Let $0 \leq t \leq s \leq T$. For any random variable ξ on Ω^t we define a shifted random variable $\xi^{s,\omega}(\tilde{\omega}) \triangleq \xi(\omega \otimes_s \tilde{\omega})$, $\forall \tilde{\omega} \in \Omega^s$ as a projection of ξ onto Ω^s along a give path ω of Ω^t , where $\omega \otimes_s \tilde{\omega}$ is the concatenation of paths ω and $\tilde{\omega}$ at time s ; see (4.1). Its discrete-time finite-state counterpart is a restriction of a binomial/trinomial tree of asset prices to one of its branches. Similarly, one can introduce shifted processes and shifted random fields, in particular, shifted controls and shifted strategies. Soner et al. [41, 44] as well as our generalization in Subsection 4.2 & 4.3 show that these shifted random objects almost surely inherit measurability and integrability.

In Proposition 4.7, we extend a result of [18] on shifted forward SDEs: For P_0^t -a.s. $\omega \in \Omega^t$, the process obtained by shifting $X^{t,x,\mu,\nu}$ solves (1.1) with parameters $(s, X_s^{t,x,\mu,\nu}(\omega), \mu^{s,\omega}, \nu^{s,\omega})$ on the probability space $(\Omega^s, \mathcal{F}_T^s, P_0^s)$. Similarly, the process obtained by shifting $(Y^{t,x,\mu,\nu}, Z^{t,x,\mu,\nu}, \underline{K}^{t,x,\mu,\nu}, \overline{K}^{t,x,\mu,\nu})$ solves (1.2) with the parameters $(s, X_s^{t,x,\mu,\nu}(\omega), \mu^{s,\omega}, \nu^{s,\omega})$ on $(\Omega^s, \mathcal{F}_T^s, P_0^s)$; see Proposition 4.8. These two propositions are also crucial in demonstrating (1.3).

iii) In constructing the ε -optimal strategies above, we use pasting of controls and strategies. Our sets of controls and strategies are closed under pasting; see Proposition 4.9 & 4.10. In the latter proof we show that an additional path-continuity of the strategies that we use to prove the dynamic programming principle is also closed under pasting.

Next, using the dynamic programming principle, the continuity of $Y^{t,x,\mu,\nu}$ in controls (μ, ν) as well as the separability of \mathbb{U}, \mathbb{V} we deduce that the value functions w_1 and w_2 are (discontinuous) viscosity solutions of the corresponding obstacle problem of fully non-linear PDEs, see Theorem 3.1.

When \mathbb{V} becomes a singleton, the zero-sum stochastic differential game degenerates into a classical stochastic control problem including only one player. In particular, when $\mathbb{U} = \{\text{all symmetric } d\text{-dimensional matrices}\}$, $b(t, x, u) = b(t, x)$ and $\sigma(t, x, u) = u$, the value function w of the optimization problem coincides with that of the second-order DRBSDEs and is related to the one in Nutz [33] via a probability transformation of strong form (5.12). Motivated by applications in financial mathematics and probabilistic numerical methods, Cheridito et al. [12] introduced second-order BSDEs. Later, Soner et al. [44] refined this notion and Soner et al. [42] related it to G -expectations of Peng [35, 34]. Quite recently Matoussi et al. [32] analyzed the second order reflected BSDEs.

The rest of the paper is organized as follows: After listing the notations used, we present two basic properties of DRBSDEs in Section 1. In Section 2, we set up the zero-sum stochastic differential games based on DRBSDEs and present a dynamic programming principle in Theorem 2.1, for priority values of both players defined via Elliott-Kalton strategies. Using the dynamic programming principle, we show in Section 3 that the priority values are (discontinuous) viscosity solutions of the corresponding obstacle problem of fully non-linear PDEs; see Theorem 3.1. In Section 4, we explore the properties of shifted processes (including measurability/integrability), shifted SDEs and pasting of controls/strategies. The contents of this section are all technical necessities in proving our main results, Theorems 2.1 and 3.1. In Section 5, we will discuss the classical stochastic control problem as a degenerate case and connect it to second order doubly reflected BSDEs. The proofs of our results are deferred to Section 6.

1.1 Notation and Preliminaries

We let \mathbb{E} denote a generic Euclidian space and let \mathbb{M} be a generic metric space with metric $\rho_{\mathbb{M}}$ and denote by $\mathcal{B}(\mathbb{M})$ the Borel σ -field on \mathbb{M} . For any $x \in \mathbb{M}$ and $\delta > 0$, $O_{\delta}(x) \triangleq \{x' \in \mathbb{M} : \rho_{\mathbb{M}}(x, x') < \delta\}$ denotes the open ball centered at x with radius δ and its closure is $\overline{O}_{\delta}(x) \triangleq \{x' \in \mathbb{M} : \rho_{\mathbb{M}}(x, x') \leq \delta\}$. For any function $\phi : \mathbb{M} \rightarrow \mathbb{R}$, we define

$$\lim_{x' \rightarrow x} \phi(x') \triangleq \lim_{n \rightarrow \infty} \uparrow \inf_{x' \in O_{1/n}(x)} \phi(x') \quad \text{and} \quad \overline{\lim}_{x' \rightarrow x} \phi(x') \triangleq \lim_{n \rightarrow \infty} \downarrow \sup_{x' \in O_{1/n}(x)} \phi(x'), \quad \forall x \in \mathbb{M}.$$

Fix $d \in \mathbb{N}$. For any $0 \leq t \leq T < \infty$, we set $\mathbb{Q}_{t,T} \triangleq ([t, T] \cap \mathbb{Q}) \cup \{T\}$, \mathbb{Q} being the rational numbers, and let $\Omega^{t,T} \triangleq \{\omega \in \mathbb{C}([t, T]; \mathbb{R}^d) : \omega(t) = 0\}$ be the canonical space over the period $[t, T]$, which is equipped with the uniform norm $\|\omega\|_{t,T} \triangleq \sup_{s \in [t, T]} |\omega(s)|$. We let $O_{\delta}(\omega) \triangleq \{\omega' \in \Omega^{t,T} : \|\omega' - \omega\|_{t,T} < \delta\}$ denote the open ball centered at $\omega \in \Omega^{t,T}$ with radius $\delta > 0$, and let $\mathcal{B}(\Omega^{t,T})$ be the correspondingly Borel σ -field of $\Omega^{t,T}$. We denote by $B^{t,T}$ the canonical process on $\Omega^{t,T}$, and by $P_0^{t,T}$ the Wiener measure on $(\Omega^{t,T}, \mathcal{B}(\Omega^{t,T}))$ under which $B^{t,T}$ is a d -dimensional Brownian motion. Let $\mathbf{F}^{t,T} = \left\{ \mathcal{F}_s^{t,T} \triangleq \sigma(B_r^{t,T}; r \in [t, s]) \right\}_{s \in [t, T]}$ be the filtration generated by $B^{t,T}$ and let $\mathcal{C}^{t,T}$ collect all *cylinder* sets in $\mathcal{F}_T^{t,T}$, i.e. $\mathcal{C}^{t,T} \triangleq \left\{ \bigcap_{i=1}^m (B_{t_i}^{t,T})^{-1}(\mathcal{E}_i) : m \in \mathbb{N}, t < t_1 < \dots < t_m \leq T, \{\mathcal{E}_i\}_{i=1}^m \subset \mathcal{B}(\mathbb{R}^d) \right\}$. It is well-known that

$$\mathcal{B}(\Omega^{t,T}) = \sigma(\mathcal{C}^{t,T}) = \sigma\left\{ (B_r^{t,T})^{-1}(\mathcal{E}) : r \in [t, T], \mathcal{E} \in \mathcal{B}(\mathbb{R}^d) \right\} = \mathcal{F}_T^{t,T}. \quad (1.4)$$

For any $\mathbf{F}^{t,T}$ -stopping time τ , we define two stochastic intervals $\llbracket t, \tau \rrbracket \triangleq \{(r, \omega) \in [t, T] \times \Omega^t : r < \tau(\omega)\}$, $\llbracket \tau, T \rrbracket \triangleq \{(r, \omega) \in [t, T] \times \Omega^t : r \geq \tau(\omega)\}$ and set $\llbracket \tau, T \rrbracket_A \triangleq \{(r, \omega) \in [t, T] \times A : r \geq \tau(\omega)\}$ for any $A \in \mathcal{F}_{\tau}^{t,T}$.

The following two results are basic, see [5] for proofs.

Lemma 1.1. *Let $0 \leq t \leq T < \infty$, for any $s \in [t, T]$, the σ -field $\mathcal{F}_s^{t,T}$ is countably generated by*

$$\mathcal{C}_s^{t,T} \triangleq \left\{ \bigcap_{i=1}^m (B_{t_i}^{t,T})^{-1}(O_{\lambda_i}(x_i)) : m \in \mathbb{N}, t_i \in \mathbb{Q} \text{ with } t \leq t_1 < \dots < t_m \leq s, x_i \in \mathbb{Q}^d, \lambda_i \in \mathbb{Q}_+ \right\}.$$

Lemma 1.2. Let $0 \leq t \leq s \leq S \leq T < \infty$. The truncation mapping $\Pi_{t,s}^{T,S} : \Omega^{t,T} \rightarrow \Omega^{s,S}$ defined by

$$(\Pi_{t,s}^{T,S}(\omega))(r) \triangleq \omega(r) - \omega(s), \quad \forall \omega \in \Omega^{t,T}, \quad \forall s \in [s, S]$$

is continuous (under uniform norms) and $\mathcal{F}_r^{t,T} / \mathcal{F}_r^{s,S}$ -measurable for any $r \in [s, S]$. Moreover, we have

$$P_0^{t,T} \left((\Pi_{t,s}^{T,S})^{-1}(A) \right) = P_0^{s,S}(A), \quad \forall A \in \mathcal{F}_S^{s,S}.$$

From now on, we fix a time horizon $T \in (0, \infty)$ and shall drop it from the above notations, i.e. $(\Omega^{t,T}, \|\cdot\|_{t,T}, B^{t,T}, \mathbf{F}^{t,T}, P_0^{t,T}, \mathcal{C}_s^{t,T}) \rightarrow (\Omega^t, \|\cdot\|_t, B^t, \mathbf{F}^t, P_0^t, \mathcal{C}_s^t)$. The expectation under P_0^t will be denoted by E_t . When $S = T$ we simply denote $\Pi_{t,s}^{T,T}$ by $\Pi_{t,s}$ in Lemma 1.2.

Given $t \in [0, T]$, we let \mathcal{P}^t denote the set of all probability measures on $(\Omega^t, \mathcal{B}(\Omega^t)) = (\Omega^t, \mathcal{F}_T^t)$ by (1.4). For any $P \in \mathcal{P}^t$, we set $\mathcal{N}^P \triangleq \{\mathcal{N} \subset \Omega^t : \mathcal{N} \subset A \text{ for some } A \in \mathcal{F}_T^t \text{ with } P(A) = 0\}$ as the collection of all P -null sets. The P -augmentation \mathbf{F}^P of \mathbf{F}^t consists of $\mathcal{F}_s^P \triangleq \sigma(\mathcal{F}_s^t \cup \mathcal{N}^P)$, $s \in [t, T]$. (In particular, we will write $\bar{\mathbf{F}}^t = \{\bar{\mathcal{F}}_s^t\}_{s \in [t, T]}$ for $\mathbf{F}^{P_0^t} = \{\mathcal{F}_s^{P_0^t}\}_{s \in [t, T]}$.) The completion of $(\Omega^t, \mathcal{F}_T^t, P)$ is the probability space $(\Omega^t, \mathcal{F}_T^P, \bar{P})$ with $\bar{P}|_{\mathcal{F}_T^t} = P$. For convenience, we will simply write P for \bar{P} .

Similar to Lemma 2.4 of [43], we have the following result:

Lemma 1.3. Let $t \in [0, T]$ and $P \in \mathcal{P}^t$.

1) For any $\xi \in \mathbb{L}^1(\mathcal{F}_T^P, P)$ and $s \in [t, T]$, $E_P[\xi | \mathcal{F}_s^P] = E_P[\xi | \mathcal{F}_s^t]$, P -a.s. Consequently, a martingale (resp. local martingale or semi-martingale) with respect to (\mathbf{F}^t, P) is also a martingale (resp. local martingale or semi-martingale) with respect to (\mathbf{F}^P, P) .

2) For any \mathbb{E} -valued, \mathbf{F}^P -adapted continuous process $\{X_s\}_{s \in [t, T]}$, there exists a unique (in sense of P -evanescence) \mathbb{E} -valued, \mathbf{F}^t -adapted continuous process $\{\tilde{X}_s\}_{s \in [t, T]}$ such that $P(\tilde{X}_s = X_s, \forall s \in [t, T]) = 1$. For any \mathbb{E} -valued, \mathbf{F}^P -progressively measurable process $\{X_s\}_{s \in [t, T]}$, there exists a unique (in $ds \times dP$ -a.s. sense) \mathbb{E} -valued, \mathbf{F}^t -progressively measurable process $\{\tilde{X}_s\}_{s \in [t, T]}$ such that $\tilde{X}_s(\omega) = X_s(\omega)$ for $ds \times dP$ -a.s. $(s, \omega) \in [t, T] \times \Omega^t$. In both cases, we call \tilde{X} the \mathbf{F}^t -version of X .

For any $p \in [1, \infty)$, $t \in [0, T]$ and $P \in \mathcal{P}^t$, we introduce some spaces of functions:

1) For any sub- σ -field \mathcal{F} of \mathcal{F}_T^P , let $\mathbb{L}^p(\mathcal{F}, \mathbb{E}, P)$ be the space of all \mathbb{E} -valued, \mathcal{F} -measurable random variables ξ such that $\|\xi\|_{\mathbb{L}^p(\mathcal{F}, P)} \triangleq \left\{ E_P[|\xi|^p] \right\}^{1/p} < \infty$.

2) For any filtration $\mathfrak{F} = \{\mathfrak{F}_s\}_{s \in [t, T]}$ on $(\Omega^t, \mathcal{F}_T^P)$, $\mathcal{P}(\mathfrak{F})$ will denote the \mathfrak{F} -progressively measurable σ -field of $[t, T] \times \Omega^t$. Let $\mathbb{C}_{\mathfrak{F}}^0([t, T], \mathbb{E}, P)$ be the space of all \mathbb{E} -valued, \mathfrak{F} -adapted processes $\{X_s\}_{s \in [t, T]}$ with P -a.s. continuous paths. We define the following subspaces of $\mathbb{C}_{\mathfrak{F}}^0([t, T], \mathbb{E}, P)$:

- $\mathbb{C}_{\mathfrak{F}}^p([t, T], \mathbb{E}, P) \triangleq \left\{ X \in \mathbb{C}_{\mathfrak{F}}^0([t, T], \mathbb{E}, P) : \|X\|_{\mathbb{C}_{\mathfrak{F}}^p([t, T], P)} \triangleq \left\{ E_P \left[\sup_{s \in [t, T]} |X_s|^p \right] \right\}^{1/p} < \infty \right\};$
- $\mathbb{C}_{\mathfrak{F}}^{\pm, p}([t, T], P) \triangleq \left\{ X \in \mathbb{C}_{\mathfrak{F}}^0([t, T], \mathbb{R}, P) : X^{\pm} \triangleq (\pm X) \vee 0 \in \mathbb{C}_{\mathfrak{F}}^p([t, T], P) \right\};$
- $\mathcal{V}_{\mathfrak{F}}([t, T], P) \triangleq \left\{ X \in \mathbb{C}_{\mathfrak{F}}^0([t, T], \mathbb{R}, P) : X \text{ has } P\text{-a.s. finite variation} \right\};$
- $\mathbb{K}_{\mathfrak{F}}([t, T], P) \triangleq \left\{ X \in \mathbb{C}_{\mathfrak{F}}^0([t, T], \mathbb{R}, P) : X_t = 0 \text{ and } X \text{ has } P\text{-a.s. increasing paths} \right\};$
- $\mathbb{K}_{\mathfrak{F}}^p([t, T], P) \triangleq \left\{ X \in \mathbb{K}_{\mathfrak{F}}([t, T], P) : E_P[X_T^p] < \infty \right\}.$

3) Let $\mathbb{H}_{\mathfrak{F}}^{p, loc}([t, T], \mathbb{E}, P)$ be the space of all \mathbb{E} -valued, \mathfrak{F} -progressively measurable processes $\{X_s\}_{s \in [t, T]}$ with $\int_t^T |X_s|^p ds < \infty$, P_0^t -a.s. And for any $\hat{p} \in [1, \infty)$, we let $\mathbb{H}_{\mathfrak{F}}^{p, \hat{p}}([t, T], \mathbb{E}, P)$ denote the space of all \mathbb{E} -valued, \mathfrak{F} -progressively measurable processes $\{X_s\}_{s \in [t, T]}$ with $\|X\|_{\mathbb{H}_{\mathfrak{F}}^{p, \hat{p}}([t, T], \mathbb{E}, P)} \triangleq \left\{ E_P \left[\left(\int_t^T |X_s|^p ds \right)^{\hat{p}/p} \right] \right\}^{1/\hat{p}} < \infty$.

Also, we set $\mathbb{G}_{\mathfrak{F}}^q([t, T], P) \triangleq \mathbb{C}_{\mathfrak{F}}^q([t, T], \mathbb{R}, P) \times \mathbb{H}_{\mathfrak{F}}^{2, q}([t, T], \mathbb{R}^d, P) \times \mathbb{K}_{\mathfrak{F}}^q([t, T], P) \times \mathbb{K}_{\mathfrak{F}}^q([t, T], P)$.

If $\mathbb{E} = \mathbb{R}$ (resp. $P = P_0^t$), we will drop it from the above notations. Moreover, we use the convention $\inf \emptyset \triangleq \infty$.

1.2 Doubly Reflected Backward Stochastic Differential Equations

Let $t \in [0, T]$. A t -parameter set $(\xi, \mathfrak{f}, \underline{L}, \overline{L})$ consists of a random variable $\xi \in \mathbb{L}^0(\overline{\mathcal{F}}_T^t)$, a function $\mathfrak{f} : [t, T] \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, and two processes $\underline{L}, \overline{L} \in \mathbb{C}_{\overline{\mathbf{F}}^t}^0([t, T])$ such that \mathfrak{f} is $\mathcal{P}(\overline{\mathbf{F}}^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$ -measurable and that $\underline{L}_T \leq \xi \leq \overline{L}_T$, P_0^t -a.s. In particular, $(\xi, \mathfrak{f}, \underline{L}, \overline{L})$ is called a (t, q) -parameter set if $\xi \in \mathbb{L}^q(\overline{\mathcal{F}}_T^t)$, $\underline{L} \in \mathbb{C}_{\overline{\mathbf{F}}^t}^{+,q}([t, T])$ and $\overline{L} \in \mathbb{C}_{\overline{\mathbf{F}}^t}^{-,q}([t, T])$.

Definition 1.1. Given $t \in [0, T]$ and a t -parameter set $(\xi, \mathfrak{f}, \underline{L}, \overline{L})$, a quadruplet $(Y, Z, \underline{K}, \overline{K}) \in \mathbb{C}_{\overline{\mathbf{F}}^t}^0([t, T]) \times \mathbb{H}_{\overline{\mathbf{F}}^t}^{2,loc}([t, T], \mathbb{R}^d) \times \mathbb{K}_{\overline{\mathbf{F}}^t}([t, T]) \times \mathbb{K}_{\overline{\mathbf{F}}^t}([t, T])$ is called a solution of the doubly reflected backward stochastic differential equation on the probability space $(\Omega^t, \overline{\mathcal{F}}_T^t, P_0^t)$ with terminal condition ξ , generator \mathfrak{f} , lower obstacle \underline{L} and upper obstacle \overline{L} (DRBSDE($P_0^t, \xi, \mathfrak{f}, \underline{L}, \overline{L}$) for short) if it holds P_0^t -a.s. that

$$\begin{cases} Y_s = \xi + \int_s^T \mathfrak{f}(r, Y_r, Z_r) dr + \underline{K}_T - \underline{K}_s - (\overline{K}_T - \overline{K}_s) - \int_s^T Z_r dB_r^t, & s \in [t, T], \\ \underline{L}_s \leq Y_s \leq \overline{L}_s, & s \in [t, T] \quad \text{and} \quad \int_t^T (Y_s - \underline{L}_s) d\underline{K}_s = \int_t^T (\overline{L}_s - Y_s) d\overline{K}_s = 0. \end{cases} \quad (1.5)$$

The last two equalities in (1.5) are known as the *flat-off* conditions corresponding to \underline{L} and \overline{L} respectively, under which the two increasing processes $\underline{K}, \overline{K}$ keep process Y between \underline{L} and \overline{L} at the minimal effort: i.e., only when Y tends to drop below \underline{L} (resp. rise above \overline{L}), \underline{K} (resp. \overline{K}) generates an upward (resp. downward) momentum.

We first have the following comparison result and a priori estimate for DRBSDEs, which generalize those in [10] and [19].

Proposition 1.1. Given $t \in [0, T]$ and two (t, q) -parameter sets $(\xi_1, \mathfrak{f}_1, \underline{L}^1, \overline{L}^1), (\xi_2, \mathfrak{f}_2, \underline{L}^2, \overline{L}^2)$ with $P_0^t(\xi_1 \leq \xi_2) = P_0^t(\underline{L}_s^1 \leq \underline{L}_s^2, \overline{L}_s^1 \leq \overline{L}_s^2, \forall s \in [t, T]) = 1$, let $(Y^i, Z^i, \underline{K}^i, \overline{K}^i) \in \mathbb{C}_{\overline{\mathbf{F}}^t}^q([t, T]) \times \mathbb{H}_{\overline{\mathbf{F}}^t}^{2,q}([t, T], \mathbb{R}^d) \times \mathbb{K}_{\overline{\mathbf{F}}^t}([t, T]) \times \mathbb{K}_{\overline{\mathbf{F}}^t}([t, T])$, $i = 1, 2$ be a solution of DRBSDE($P_0^t, \xi_i, \mathfrak{f}_i, \underline{L}^i, \overline{L}^i$). For either $i = 1$ or $i = 2$, if \mathfrak{f}_i is Lipschitz continuous in (y, z) : i.e. for some $\gamma > 0$, it holds for $ds \times dP_0^t$ -a.s. $(s, \omega) \in [t, T] \times \Omega^t$ that

$$|\mathfrak{f}_i(s, \omega, y, z) - \mathfrak{f}_i(s, \omega, y', z')| \leq \gamma(|y - y'| + |z - z'|), \quad \forall y, y' \in \mathbb{R}, \quad \forall z, z' \in \mathbb{R}^d, \quad (1.6)$$

and if $\mathfrak{f}_1(s, Y_s^{3-i}, Z_s^{3-i}) \leq \mathfrak{f}_2(s, Y_s^{3-i}, Z_s^{3-i})$, $ds \times dP_0^t$ -a.s., then it holds P_0^t -a.s. that $Y_s^1 \leq Y_s^2$ for any $s \in [t, T]$.

Proposition 1.2. Given $t \in [0, T]$ and two (t, q) -parameter sets $(\xi_1, \mathfrak{f}_1, \underline{L}^1, \overline{L}^1), (\xi_2, \mathfrak{f}_2, \underline{L}^2, \overline{L}^2)$ with $P_0^t(\underline{L}_s^1 = \underline{L}_s^2, \overline{L}_s^1 = \overline{L}_s^2, \forall s \in [t, T]) = 1$, let $(Y^i, Z^i, \underline{K}^i, \overline{K}^i) \in \mathbb{C}_{\overline{\mathbf{F}}^t}^q([t, T]) \times \mathbb{H}_{\overline{\mathbf{F}}^t}^{2,q}([t, T], \mathbb{R}^d) \times \mathbb{K}_{\overline{\mathbf{F}}^t}([t, T]) \times \mathbb{K}_{\overline{\mathbf{F}}^t}([t, T])$, $i = 1, 2$ be a solution of DRBSDE($P_0^t, \xi_i, \mathfrak{f}_i, \underline{L}^i, \overline{L}^i$). If \mathfrak{f}_1 satisfies (1.6), then for any $\varpi \in (1, q]$

$$E_t \left[\sup_{s \in [t, T]} |Y_s^1 - Y_s^2|^\varpi \right] \leq C(T, \varpi, \gamma) \left\{ E_t[|\xi_1 - \xi_2|^\varpi] + E_t \left[\left(\int_t^T \mathfrak{f}_1(r, Y_r^2, Z_r^2) - \mathfrak{f}_2(r, Y_r^2, Z_r^2) dr \right)^\varpi \right] \right\}. \quad (1.7)$$

Given a (t, q) -parameter set $(\xi, \mathfrak{f}, \underline{L}, \overline{L})$ such that \mathfrak{f} is Lipschitz continuous in (y, z) . If $E_t \left[\left(\int_t^T |\mathfrak{f}(s, 0, 0)| ds \right)^q \right] < \infty$ and if $P_0^t(\underline{L}_s < \overline{L}_s, \forall s \in [t, T]) = 1$, then we know from Theorem 4.1 of [15] that the DRBSDE($P_0^t, \xi, \mathfrak{f}, \underline{L}, \overline{L}$) admits a unique solution $(Y, Z, \underline{K}, \overline{K}) \in \mathbb{G}_{\overline{\mathbf{F}}^t}^q([t, T])$.

2 Stochastic Differential Games with Square-Integrable Controls

Let $(\mathbb{U}, \rho_{\mathbb{U}})$ and $(\mathbb{V}, \rho_{\mathbb{V}})$ be two separable metric spaces, whose Borel- σ -fields will be denoted by $\mathcal{B}(\mathbb{U})$ and $\mathcal{B}(\mathbb{V})$ respectively. For some $u_0 \in \mathbb{U}$ and $v_0 \in \mathbb{V}$, we define

$$[u]_{\mathbb{U}} \triangleq \rho_{\mathbb{U}}(u, u_0), \quad \forall u \in \mathbb{U} \quad \text{and} \quad [v]_{\mathbb{V}} \triangleq \rho_{\mathbb{V}}(v, v_0), \quad \forall v \in \mathbb{V}.$$

Fix a non-empty $\mathbb{U}_0 \subset \mathbb{U}$ and a non-empty $\mathbb{V}_0 \subset \mathbb{V}$. We shall study a zero-sum stochastic differential game between two players, player I and player II, who choose square-integrable \mathbb{U} -valued controls and \mathbb{V} -valued controls respectively to compete:

Definition 2.1. Given $t \in [0, T]$, an admissible control process $\mu = \{\mu_r\}_{r \in [t, T]}$ for player I over period $[t, T]$ is a \mathbb{U} -valued, \mathbf{F}^t -progressively measurable process such that $\mu_r \in \mathbb{U}_0$, $dr \times dP_0^t$ -a.s. and that $E_t \int_t^T [\mu_r]_{\mathbb{U}}^2 dr < \infty$. Admissible control processes for player II over period $[t, T]$ are defined similarly. The set of all admissible controls for player I (resp. II) over period $[t, T]$ is denoted by \mathcal{U}^t (resp. \mathcal{V}^t).

Our zero-sum stochastic differential game is formulated via a (decoupled) SDE–DRBSDE system with the following parameters: Fix $k \in \mathbb{N}$, $\gamma > 0$ and $q \in (1, 2]$.

1) Let $b : [0, T] \times \mathbb{R}^k \times \mathbb{U} \times \mathbb{V} \rightarrow \mathbb{R}^k$ be a $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{U}) \otimes \mathcal{B}(\mathbb{V}) / \mathcal{B}(\mathbb{R}^k)$ -measurable function and let $\sigma : [0, T] \times \mathbb{R}^k \times \mathbb{U} \times \mathbb{V} \rightarrow \mathbb{R}^{k \times d}$ be a $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{U}) \otimes \mathcal{B}(\mathbb{V}) / \mathcal{B}(\mathbb{R}^{k \times d})$ -measurable function such that for any $(t, u, v) \in [0, T] \times \mathbb{U} \times \mathbb{V}$ and any $x, x' \in \mathbb{R}^k$

$$|b(t, 0, u, v)| + |\sigma(t, 0, u, v)| \leq \gamma(1 + [u]_{\mathbb{U}} + [v]_{\mathbb{V}}) \quad (2.1)$$

$$\text{and} \quad |b(t, x, u, v) - b(t, x', u, v)| + |\sigma(t, x, u, v) - \sigma(t, x', u, v)| \leq \gamma|x - x'|; \quad (2.2)$$

2) Let $\underline{l}, \bar{l} : [0, T] \times \mathbb{R}^k \rightarrow \mathbb{R}$ be two continuous functions such that $\underline{l}(t, x) < \bar{l}(t, x)$, $\forall (t, x) \in [0, T] \times \mathbb{R}^k$ and that

$$|\underline{l}(t, x) - \underline{l}(t, x')| \vee |\bar{l}(t, x) - \bar{l}(t, x')| \leq \gamma|x - x'|^{2/q}, \quad \forall t \in [0, T], \quad \forall x, x' \in \mathbb{R}^k; \quad (2.3)$$

3) Let $h : \mathbb{R}^k \rightarrow \mathbb{R}$ be a $2/q$ -Hölder continuous function with coefficient γ such that $\underline{l}(T, x) \leq h(x) \leq \bar{l}(T, x)$, $\forall x \in \mathbb{R}^k$;

4) Let $f : [0, T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{U} \times \mathbb{V} \rightarrow \mathbb{R}$ be $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{U}) \otimes \mathcal{B}(\mathbb{V}) / \mathcal{B}(\mathbb{R})$ -measurable function such that for any $(t, u, v) \in [0, T] \times \mathbb{U} \times \mathbb{V}$ and any $(x, y, z), (x', y', z') \in \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d$

$$|f(t, 0, 0, 0, u, v)| \leq \gamma(1 + [u]_{\mathbb{U}}^{2/q} + [v]_{\mathbb{V}}^{2/q}) \quad (2.4)$$

$$\text{and} \quad |f(t, x, y, z, u, v) - f(t, x', y', z', u, v)| \leq \gamma(|x - x'|^{2/q} + |y - y'| + |z - z'|). \quad (2.5)$$

For any $\lambda \geq 0$, we let c_λ denote a generic constant, depending on $\lambda, T, q, \gamma, \underline{l}_* \triangleq \sup_{s \in [t, T]} |\underline{l}(s, 0)|$ and $\bar{l}_* \triangleq \sup_{s \in [t, T]} |\bar{l}(s, 0)|$, whose form may vary from line to line. (In particular, c_0 stands for a generic constant depending on $T, q, \gamma, \underline{l}_*$ and \bar{l}_* .)

Fix $t \in [0, T]$. Assume that when player I (resp. II) select admissible control $\mu \in \mathcal{U}^t$ (resp. $\nu \in \mathcal{V}^t$), the corresponding state process starting from time t at point $x \in \mathbb{R}^k$ is driven by the SDE (1.1) on the probability space $(\Omega^t, \bar{\mathcal{F}}_T^t, P_0^t)$. Clearly, the measurability of b and σ implies that for any $x' \in \mathbb{R}^k$, both $\{b(s, x', \mu_s, \nu_s)\}_{s \in [t, T]}$ and $\{\sigma(s, x', \mu_s, \nu_s)\}_{s \in [t, T]}$ are \mathbf{F}^t -progressively measurable processes. Also, we see from (2.2) that for any $(s, \omega) \in [t, T] \times \Omega^t$, both $b(s, \cdot, \mu_s(\omega), \nu_s(\omega))$ and $\sigma(s, \cdot, \mu_s(\omega), \nu_s(\omega))$ are Lipschitz with coefficient γ . Since

$$E_t \int_t^T (|b(s, 0, \mu_s, \nu_s)|^2 + |\sigma(s, 0, \mu_s, \nu_s)|^2) ds \leq c_0 + c_0 E_t \int_t^T ([\mu_s]_{\mathbb{U}}^2 + [\nu_s]_{\mathbb{V}}^2) ds < \infty \quad (2.6)$$

by (2.1), it is well-known (see e.g. Theorem 2.5.7 of [30]) that (1.1) admits a unique solution $\{X_s^{t, x, \mu, \nu}\}_{s \in [t, T]} \in \mathbb{C}_{\mathbf{F}^t}^2([t, T], \mathbb{R}^k)$.

Applying Theorem 2.5.9 of [30] with $(\xi_s, \tilde{\xi}_s, \tilde{b}_s(0), \tilde{\sigma}_s(0)) \equiv (x, 0, 0, 0)$ (thus $\tilde{x}_s \equiv 0$ therein) and using (2.1) yield

$$E_t \left[\sup_{s \in [t, T]} |X_s^{t, x, \mu, \nu}|^2 \right] \leq c_0 \left(1 + |x|^2 + E_t \int_t^T ([\mu_s]_{\mathbb{U}}^2 + [\nu_s]_{\mathbb{V}}^2) ds \right) < \infty. \quad (2.7)$$

2.1 Continuous Dependence Results

Lemma 2.1. Let $\varpi \in [1, 2]$, $t \in [0, T]$ and $(x, \mu, \nu) \in \mathbb{R}^k \times \mathcal{U}^t \times \mathcal{V}^t$.

$$(1) \text{ For any } s \in [t, T], \quad E_t \left[\sup_{r \in [t, s]} |X_r^{t, x, \mu, \nu} - x|^2 \right] \leq c_0(1 + |x|^2)(s - t) + c_0 E_t \int_t^s ([\mu_r]_{\mathbb{U}}^2 + [\nu_r]_{\mathbb{V}}^2) dr. \quad (2.8)$$

$$(2) \text{ For any } x' \in \mathbb{R}^k, \quad E_t \left[\sup_{s \in [t, T]} |X_s^{t, x, \mu, \nu} - X_s^{t, x', \mu, \nu}|^\varpi \right] \leq c_\varpi |x - x'|^\varpi. \quad (2.9)$$

(3) If b and σ are λ -Hölder continuous in u for some $\lambda \in (0, 1]$, i.e., for any $(\bar{t}, \bar{x}, \bar{v}) \in [0, T] \times \mathbb{R}^k \times \mathbb{V}$ and $\bar{u}_1, \bar{u}_2 \in \mathbb{U}$

$$|b(\bar{t}, \bar{x}, \bar{u}_1, \bar{v}) - b(\bar{t}, \bar{x}, \bar{u}_2, \bar{v})| + |\sigma(\bar{t}, \bar{x}, \bar{u}_1, \bar{v}) - \sigma(\bar{t}, \bar{x}, \bar{u}_2, \bar{v})| \leq \gamma \rho_{\mathbb{U}}^{\lambda}(\bar{u}_1, \bar{u}_2), \quad (2.10)$$

then for any $\mu' \in \mathcal{U}^t$

$$E_t \left[\sup_{s \in [t, T]} |X_s^{t, x, \mu, \nu} - X_s^{t, x, \mu', \nu}|^{\varpi} \right] \leq c_{\varpi} E_t \left[\left(\int_t^T \rho_{\mathbb{U}}^{2\lambda}(\mu_r, \mu'_r) dr \right)^{\varpi/2} \right]. \quad (2.11)$$

Similarly, if b and σ are λ -Hölder continuous in u for some $\lambda \in (0, 1]$, i.e., for any $(\bar{t}, \bar{x}, \bar{u}) \in [0, T] \times \mathbb{R}^k \times \mathbb{U}$ and $\bar{v}_1, \bar{v}_2 \in \mathbb{V}$

$$|b(\bar{t}, \bar{x}, \bar{u}, \bar{v}_1) - b(\bar{t}, \bar{x}, \bar{u}, \bar{v}_2)| + |\sigma(\bar{t}, \bar{x}, \bar{u}, \bar{v}_1) - \sigma(\bar{t}, \bar{x}, \bar{u}, \bar{v}_2)| \leq \gamma \rho_{\mathbb{V}}^{\lambda}(\bar{v}_1, \bar{v}_2), \quad (2.12)$$

then for any $\nu' \in \mathcal{V}^t$

$$E_t \left[\sup_{s \in [t, T]} |X_s^{t, x, \mu, \nu} - X_s^{t, x, \mu, \nu'}|^{\varpi} \right] \leq c_{\varpi} E_t \left[\left(\int_t^T \rho_{\mathbb{V}}^{2\lambda}(\nu_r, \nu'_r) dr \right)^{\varpi/2} \right]. \quad (2.13)$$

By Lemma 1.3 (2), $X^{t, x, \mu, \nu}$ admits a unique \mathbf{F}^t -version $\tilde{X}^{t, x, \mu, \nu}$, which clearly belongs to $\mathbb{C}_{\mathbf{F}^t}^2([t, T], \mathbb{R}^k)$ and also satisfies (2.7), (2.8), (2.9), (2.11) and (2.13).

If $(\tilde{\mu}, \tilde{\nu})$ is another pair of $\mathcal{U}^t \times \mathcal{V}^t$ such that $(\mu, \nu) = (\tilde{\mu}, \tilde{\nu})$ $dr \times dP_0^t$ -a.s. on $\llbracket t, \tau \rrbracket$ for some \mathbf{F}^t -stopping time τ , then both $\{X_{\tau \wedge s}^{t, x, \mu, \nu}\}_{s \in [t, T]}$ and $\{X_{\tau \wedge s}^{t, x, \tilde{\mu}, \tilde{\nu}}\}_{s \in [t, T]}$ satisfy the same SDE:

$$X_s = x + \int_t^s b_{\tau}(r, X_r) dr + \int_t^s \sigma_{\tau}(r, X_r) dB_r^t, \quad s \in [t, T], \quad (2.14)$$

with $b_{\tau}(r, \omega, x) \triangleq \mathbf{1}_{\{r < \tau(\omega)\}} b(r, x, \mu_r(\omega), \nu_r(\omega))$ and $\sigma_{\tau}(r, \omega, x) \triangleq \mathbf{1}_{\{r < \tau(\omega)\}} \sigma(r, x, \mu_r(\omega), \nu_r(\omega))$, $\forall (r, \omega, x) \in [t, T] \times \Omega^t \times \mathbb{R}^k$. Clearly, for any $x \in \mathbb{R}^k$ both $b_{\tau}(\cdot, \cdot, x)$ and $\sigma_{\tau}(\cdot, \cdot, x)$ are \mathbf{F}^t -progressively measurable processes, and for any $(r, \omega) \in [t, T] \times \Omega^t$ both $b_{\tau}(r, \omega, \cdot)$ and $\sigma_{\tau}(r, \omega, \cdot)$ are Lipschitz continuous with coefficient γ . Thus (2.14) has a unique solution in $\mathbb{C}_{\mathbf{F}^t}^2([t, T], \mathbb{R}^k)$, i.e.

$$P_0^t(X_{\tau \wedge s}^{t, x, \mu, \nu} = X_{\tau \wedge s}^{t, x, \tilde{\mu}, \tilde{\nu}}, \quad \forall s \in [t, T]) = 1. \quad (2.15)$$

Let Θ stand for the quadruplet (t, x, μ, ν) . By the continuity of \underline{l} and \bar{l} , $\underline{L}_s^{\Theta} \triangleq \underline{l}(s, \tilde{X}_s^{\Theta})$ and $\bar{L}_s^{\Theta} \triangleq \bar{l}(s, \tilde{X}_s^{\Theta})$, $s \in [t, T]$ are two real-valued, \mathbf{F}^t -adapted continuous processes such that $\underline{L}_s^{\Theta} < \bar{L}_s^{\Theta}$, $\forall s \in [t, T]$.

Given an \mathbf{F}^t -stopping time τ , the measurability of $(f, \tilde{X}^{\Theta}, \mu, \nu)$ and (2.5) imply that

$$f_{\tau}^{\Theta}(s, \omega, y, z) \triangleq \mathbf{1}_{\{s < \tau(\omega)\}} f\left(s, \tilde{X}_s^{\Theta}(\omega), y, z, \mu_s(\omega), \nu_s(\omega)\right), \quad \forall (s, \omega, y, z) \in [t, T] \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d$$

is a $\mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) / \mathcal{B}(\mathbb{R})$ -measurable function that is Lipschitz continuous in (y, z) with coefficient γ . And one can deduce from (2.3), (2.4), (2.5), Hölder's inequality and (2.7) that

$$E_t \left[\sup_{s \in [t, T]} |\underline{L}_{\tau \wedge s}^{\Theta}|^q + \sup_{s \in [t, T]} |\bar{L}_{\tau \wedge s}^{\Theta}|^q + \left(\int_t^T |f_{\tau}^{\Theta}(s, 0, 0)| ds \right)^q \right] \leq c_0 + c_0 E_t \left[\sup_{s \in [t, T]} |\tilde{X}_s^{\Theta}|^2 + \int_t^T ([\mu_s]_{\mathbb{U}}^2 + [\nu_s]_{\mathbb{V}}^2) ds \right] < \infty. \quad (2.16)$$

Thus, for any \mathcal{F}_{τ}^t -measurable random variable ξ with $\underline{L}_{\tau}^{\Theta} \leq \xi \leq \bar{L}_{\tau}^{\Theta}$, P_0^t -a.s., it follows that $E_t[|\xi|^q] < \infty$, i.e. $\xi \in \mathbb{L}^q(\mathcal{F}_{\tau}^t)$. Then Theorem 4.1 of [15] shows that the DRBSDE($P_0^t, \xi, f_{\tau}^{\Theta}, \underline{L}_{\tau \wedge \cdot}^{\Theta}, \bar{L}_{\tau \wedge \cdot}^{\Theta}$) admits a unique solution $(Y^{\Theta}(\tau, \xi), Z^{\Theta}(\tau, \xi), \underline{K}^{\Theta}(\tau, \xi), \bar{K}^{\Theta}(\tau, \xi)) \in \mathbb{G}_{\mathbf{F}^t}^q([t, T])$. Clearly, its \mathbf{F}^t -version $(\tilde{Y}^{\Theta}(\tau, \xi), \tilde{Z}^{\Theta}(\tau, \xi), \tilde{\underline{K}}^{\Theta}(\tau, \xi), \tilde{\bar{K}}^{\Theta}(\tau, \xi))$ by Lemma 1.3 (2) belongs to $\mathbb{G}_{\mathbf{F}^t}^q([t, T])$. As $\mathcal{F}_t^t = \{\emptyset, \Omega^t\}$, $\tilde{Y}_t^{\Theta}(\tau, \xi)$ is a constant.

Given another \mathbf{F}^t -stopping time ζ such that $\zeta \leq \tau$, P_0^t -a.s., one can easily show that $\left\{ \left(\tilde{Y}_{\zeta \wedge s}^\Theta(\tau, \xi), \mathbf{1}_{\{s < \zeta\}} \tilde{Z}_s^\Theta(\tau, \xi), \tilde{K}_{\zeta \wedge s}^\Theta(\tau, \xi), \tilde{\bar{K}}_{\zeta \wedge s}^\Theta(\tau, \xi) \right) \right\}_{s \in [t, T]} \in \mathbb{G}_{\mathbf{F}^t}^q([t, T])$ solves the DRBSDE $\left(P_0^t, \tilde{Y}_\zeta^\Theta(\tau, \xi), f_\zeta^\Theta, \underline{L}_{\zeta \wedge \cdot}^\Theta, \bar{L}_{\zeta \wedge \cdot}^\Theta \right)$. To wit, we have

$$\begin{aligned} & \left(\tilde{Y}_s^\Theta(\zeta, \tilde{Y}_\zeta^\Theta(\tau, \xi)), \tilde{Z}_s^\Theta(\zeta, \tilde{Y}_\zeta^\Theta(\tau, \xi)), \tilde{K}_s^\Theta(\zeta, \tilde{Y}_\zeta^\Theta(\tau, \xi)), \tilde{\bar{K}}_s^\Theta(\zeta, \tilde{Y}_\zeta^\Theta(\tau, \xi)) \right) \\ &= \left(\tilde{Y}_{\zeta \wedge s}^\Theta(\tau, \xi), \mathbf{1}_{\{s < \zeta\}} \tilde{Z}_s^\Theta(\tau, \xi), \tilde{K}_{\zeta \wedge s}^\Theta(\tau, \xi), \tilde{\bar{K}}_{\zeta \wedge s}^\Theta(\tau, \xi) \right), \quad s \in [t, T]. \end{aligned} \quad (2.17)$$

The continuity of functions h implies that $h(\tilde{X}_T^\Theta)$ is a real-valued, \mathcal{F}_T^t -measurable random variables such that $\underline{L}_T^\Theta = \underline{l}(T, \tilde{X}_T^\Theta) \leq h(\tilde{X}_T^\Theta) \leq \bar{l}(T, \tilde{X}_T^\Theta) = \bar{L}_T^\Theta$. Hence, we can use (1.5) to obtain that

$$\underline{l}(t, x) = \underline{l}(t, \tilde{X}_t^\Theta) \leq Y_t^\Theta(T, h(\tilde{X}_T^\Theta)) \leq \bar{l}(t, \tilde{X}_t^\Theta) = \bar{l}(t, x), \quad P_0^t - a.s.,$$

which leads to that

$$\underline{l}(t, x) \leq \tilde{Y}_t^\Theta(T, h(\tilde{X}_T^\Theta)) \leq \bar{l}(t, x). \quad (2.18)$$

Inspired by Proposition 6.1 (ii) of [10], we have the following a priori estimates about the dependence of $Y_s^{t, x, \mu, \nu}(T, h(\tilde{X}_T^{t, x, \mu, \nu}))$ on initial state x and on controls (μ, ν) .

Lemma 2.2. *Let $\varpi \in (1, q]$, $t \in [0, T]$ and $(x, \mu, \nu) \in \mathbb{R}^k \times \mathcal{U}^t \times \mathcal{V}^t$.*

$$(1) \text{ For any } x' \in \mathbb{R}^k, \quad E_t \left[\sup_{s \in [t, T]} \left| Y_s^{t, x, \mu, \nu}(T, h(\tilde{X}_T^{t, x, \mu, \nu})) - Y_s^{t, x', \mu, \nu}(T, h(\tilde{X}_T^{t, x', \mu, \nu})) \right|^\varpi \right] \leq c_\varpi |x - x'|^\frac{2\varpi}{q}. \quad (2.19)$$

(2) Let \underline{l}, \bar{l} and h satisfy

$$|\underline{l}(t, x) - \underline{l}(t, x')| \vee |\bar{l}(t, x) - \bar{l}(t, x')| \vee |h(x) - h(x')| \leq \gamma \psi(|x - x'|), \quad \forall t \in [0, T], \quad \forall x, x' \in \mathbb{R}^k \quad (2.20)$$

for an increasing $\mathbb{C}^2(\mathbb{R}_+)$ function ψ such that for some $0 < R_1 < 1 < R_2$

$$\psi(a) = \frac{1}{2}a^2 \text{ if } a \in [0, R_1], \quad \psi(a) \leq a^{2/q} \text{ if } a \in (R_1, R_2), \quad \text{and } \psi(a) = a^{2/q} \text{ if } a > R_2.$$

(Clearly, $\psi(a) \leq a^{2/q}$ for any $a \geq 0$. So (2.20) implies (2.3) and the $2/q$ -Hölder continuity of h .)

Let $\lambda \in (0, 1]$. If b, σ are λ -Hölder continuous in u (see (2.10)) and if f is 2λ -Hölder continuous in u , i.e., for any $(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{v}) \in [0, T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{V}$ and $\bar{u}_1, \bar{u}_2 \in \mathbb{U}$

$$|f(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}_1, \bar{v}) - f(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}_2, \bar{v})| \leq \gamma \rho_{\mathbb{U}}^{2\lambda}(\bar{u}_1, \bar{u}_2), \quad (2.21)$$

then for any $\mu' \in \mathcal{U}^t$

$$\begin{aligned} & E_t \left[\sup_{s \in [t, T]} \left| Y_s^{t, x, \mu, \nu}(T, h(\tilde{X}_T^{t, x, \mu, \nu})) - Y_s^{t, x, \mu', \nu}(T, h(\tilde{X}_T^{t, x, \mu', \nu})) \right|^\varpi \right] \\ & \leq c_\varpi \kappa_\psi^\varpi \left\{ E_t \left[\left(\int_t^T \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s) ds \right)^\frac{\varpi}{2} \right] + E_t \left[\left(\int_t^T \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s) ds \right)^\varpi \right] \right\}, \end{aligned} \quad (2.22)$$

where $\kappa_\psi \triangleq \left(2 + R_1^{-1} \sup_{a \in [R_1, R_2]} 1 \vee \psi'(a) + \sup_{a \in [R_1, R_2]} |\psi''(a)| \right) R_2^{2-\frac{2}{q}}$.

Similarly, if b, σ are λ -Hölder continuous in v (see (2.12)) and if f is additionally 2λ -Hölder continuous in v , i.e., for any $(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}) \in [0, T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{U}$ and $\bar{v}_1, \bar{v}_2 \in \mathbb{V}$

$$|f(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}_1) - f(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}_2)| \leq \gamma \rho_{\mathbb{V}}^{2\lambda}(\bar{v}_1, \bar{v}_2), \quad (2.23)$$

then for any $\nu' \in \mathcal{V}^t$

$$\begin{aligned} & E_t \left[\sup_{s \in [t, T]} \left| Y_s^{t, x, \mu, \nu}(T, h(\tilde{X}_T^{t, x, \mu, \nu})) - Y_s^{t, x, \mu, \nu'}(T, h(\tilde{X}_T^{t, x, \mu, \nu'})) \right|^\varpi \right] \\ & \leq c_\varpi \kappa_\psi^\varpi \left\{ E_t \left[\left(\int_t^T \rho_{\mathbb{V}}^{2\lambda}(\nu'_s, \nu_s) ds \right)^\frac{\varpi}{2} \right] + E_t \left[\left(\int_t^T \rho_{\mathbb{V}}^{2\lambda}(\nu'_s, \nu_s) ds \right)^\varpi \right] \right\}. \end{aligned} \quad (2.24)$$

2.2 Definition of the value functions and the Dynamic Programming Principle

Now, we are ready to introduce values of the zero-sum stochastic differential games via the following notion of admissible strategies.

Definition 2.2. *Given $t \in [0, T]$, an admissible strategy α for player I over period $[t, T]$ is a \mathbb{U} -valued function α on $[t, T] \times \Omega^t \times \mathbb{V}$ that is $\mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{V}) / \mathcal{B}(\mathbb{U})$ -measurable and satisfies: (i) $\alpha(r, \mathbb{V}_0) \subset \mathbb{U}_0$, $dr \times dP_0^t$ -a.s. (ii) For a $\kappa > 0$ and a non-negative measurable process Ψ on $(\Omega^t, \mathcal{F}_T^t)$ with $E_t \int_t^T \Psi_r^2 dr < \infty$, it holds $dr \times dP_0^t$ -a.s. that*

$$[\alpha(r, \omega, v)]_{\mathbb{U}} \leq \Psi_r(\omega) + \kappa[v]_{\mathbb{V}}, \quad \forall v \in \mathbb{V}. \quad (2.25)$$

Admissible strategies $\beta : [t, T] \times \Omega^t \times \mathbb{U} \rightarrow \mathbb{V}$ for player II over period $[t, T]$ are defined similarly. The set of all admissible strategies for player I (resp. II) on $[t, T]$ is denoted by \mathcal{A}^t (resp. \mathfrak{B}^t).

Given $t \in [0, T]$, an admissible strategy $\alpha \in \mathcal{A}^t$ induces a mapping $\alpha\langle \cdot \rangle : \mathcal{V}^t \rightarrow \mathcal{U}^t$ by

$$(\alpha\langle \nu \rangle)_r(\omega) \triangleq \alpha(r, \omega, \nu_r(\omega)), \quad \forall \nu \in \mathcal{V}^t, (r, \omega) \in [t, T] \times \Omega^t.$$

To see this, let $\nu \in \mathcal{V}^t$. Clearly, $\alpha\langle \nu \rangle$ is a \mathbb{U} -valued, \mathbf{F}^t -progressively measurable process. Since $\{(r, \omega) \in [t, T] \times \Omega^t : \nu_r(\omega) \in \mathbb{V}_0\} \cap \{(r, \omega) \in [t, T] \times \Omega^t : \alpha(r, \omega, \mathbb{V}_0) \subset \mathbb{U}_0\} \subset \{(r, \omega) \in [t, T] \times \Omega^t : (\alpha\langle \nu \rangle)_r(\omega) \in \mathbb{U}_0\}$, it holds $dr \times dP_0^t$ -a.s. that $\alpha\langle \nu \rangle \in \mathbb{U}_0$. On the other hand, one can deduce that

$$E_t \int_t^T [(\alpha\langle \nu \rangle)_r]_{\mathbb{U}}^2 dr \leq 2E_t \int_t^T \Psi_r^2 dr + 2\kappa^2 E_t \int_t^T [\nu_r]_{\mathbb{V}}^2 dr < \infty.$$

Thus, $\alpha\langle \nu \rangle \in \mathcal{U}^t$. If $\nu^1 \in \mathcal{V}^t$ is equal to $\nu^2 \in \mathcal{V}^t$, $dr \times dP_0^t$ -a.s. on $[[t, \tau[$ for any \mathbf{F}^t -stopping time τ , then $\alpha\langle \nu^1 \rangle = \alpha\langle \nu^2 \rangle$, $dr \times dP_0^t$ -a.s. on $[[t, \tau[$. So $\alpha\langle \cdot \rangle$ is exactly an Elliott–Kalton strategy considered in e.g. [18]. Similarly, any $\beta \in \mathfrak{B}^t$ gives rise to a mapping $\beta\langle \cdot \rangle : \mathcal{U}^t \rightarrow \mathcal{V}^t$.

Definition 2.3. *Given $t \in [0, T]$, an \mathcal{A}^t -strategy α is said to be of $\hat{\mathcal{A}}^t$ if for any $\varepsilon > 0$, there exist a $\delta > 0$ and a closed subset F of Ω^t with $P_0^t(F) > 1 - \varepsilon$ such that for any $\omega, \omega' \in F$ with $\|\omega - \omega'\|_t < \delta$*

$$\sup_{r \in [t, T]} \sup_{v \in \mathbb{V}} \rho_{\mathbb{U}}(\alpha(r, \omega, v), \alpha(r, \omega', v)) < \varepsilon. \quad (2.26)$$

We define $\hat{\mathfrak{B}}^t \subset \mathfrak{B}^t$ similarly.

For any $(t, x) \in [0, T] \times \mathbb{R}^k$, we define

$$w_1(t, x) \triangleq \inf_{\beta \in \mathfrak{B}^t} \sup_{\mu \in \mathcal{U}^t} \tilde{Y}_t^{t, x, \mu, \beta\langle \mu \rangle} \left(T, h \left(\tilde{X}_T^{t, x, \mu, \beta\langle \mu \rangle} \right) \right) \quad \text{and} \quad \hat{w}_1(t, x) \triangleq \inf_{\beta \in \hat{\mathfrak{B}}^t} \sup_{\mu \in \mathcal{U}^t} \tilde{Y}_t^{t, x, \mu, \beta\langle \mu \rangle} \left(T, h \left(\tilde{X}_T^{t, x, \mu, \beta\langle \mu \rangle} \right) \right)$$

as player I's *priority value* and *intrinsic priority value* of the zero-sum stochastic differential game that starts from time t and state x . Correspondingly, we define

$$w_2(t, x) \triangleq \sup_{\alpha \in \mathcal{A}^t} \inf_{\nu \in \mathcal{V}^t} \tilde{Y}_t^{t, x, \alpha\langle \nu \rangle, \nu} \left(T, h \left(\tilde{X}_T^{t, x, \alpha\langle \nu \rangle, \nu} \right) \right) \quad \text{and} \quad \hat{w}_2(t, x) \triangleq \sup_{\alpha \in \hat{\mathcal{A}}^t} \inf_{\nu \in \mathcal{V}^t} \tilde{Y}_t^{t, x, \alpha\langle \nu \rangle, \nu} \left(T, h \left(\tilde{X}_T^{t, x, \alpha\langle \nu \rangle, \nu} \right) \right)$$

as player II's *priority value* and *intrinsic priority value* of the zero-sum stochastic differential game that starts from time t and state x . By (2.18), one has

$$\underline{l}(t, x) \leq w_1(t, x) \leq \hat{w}_1(t, x) \leq \bar{l}(t, x) \quad \text{and} \quad \underline{l}(t, x) \leq \hat{w}_2(t, x) \leq w_2(t, x) \leq \bar{l}(t, x). \quad (2.27)$$

The two obstacle functions \underline{l}, \bar{l} as well as the DRBSDE structure prevent the value functions from taking $\pm\infty$ values. The values $w_1(t, x)$ and $\hat{w}_1(t, x)$, otherwise, might blow up unless we impose additional integrability conditions on \mathcal{U}^t and \mathfrak{B}^t analogous to e.g. Assumption 5.7 of [44].

Remark 2.1. Given $t \in [0, T]$, we can regard $\mu \in \mathcal{U}^t$ as a member of \mathcal{A}^t since

$$\alpha^\mu(r, \omega, v) \triangleq \mu_r(\omega), \quad \forall (r, \omega, v) \in [t, T] \times \Omega^t \times \mathbb{V}$$

is clearly a $\mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{V}) \rightarrow \mathcal{B}(\mathbb{U})$ -measurable function such that $\alpha^\mu(r, \mathbb{V}_0) = \mu_r \in \mathbb{U}_0$, $dr \times dP_0^t$ -a.s. and that (2.25) holds for $\Psi^\alpha = [\mu]_{\mathbb{U}}$ and any $\kappa_\alpha > 0$. Similarly, \mathcal{V}^t can be embedded into \mathfrak{B}^t . Then it follows that

$$w_1(t, x) \leq \inf_{\nu \in \mathcal{V}^t} \sup_{\mu \in \mathcal{U}^t} \tilde{Y}_t^{t, x, \mu, \nu} \left(T, h(\tilde{X}_T^{t, x, \mu, \nu}) \right) \quad \text{and} \quad w_2(t, x) \geq \sup_{\mu \in \mathcal{U}^t} \inf_{\nu \in \mathcal{V}^t} \tilde{Y}_t^{t, x, \mu, \nu} \left(T, h(\tilde{X}_T^{t, x, \mu, \nu}) \right).$$

However, the fact that $\sup_{\mu \in \mathcal{U}^t} \inf_{\nu \in \mathcal{V}^t} \tilde{Y}_t^{t, x, \mu, \nu} \left(T, h(\tilde{X}_T^{t, x, \mu, \nu}) \right) \leq \inf_{\nu \in \mathcal{V}^t} \sup_{\mu \in \mathcal{U}^t} \tilde{Y}_t^{t, x, \mu, \nu} \left(T, h(\tilde{X}_T^{t, x, \mu, \nu}) \right)$ does not necessarily imply that $w_2(t, x) \leq w_1(t, x)$.

Let $t \in [0, T]$ and $x_1, x_2 \in \mathbb{R}^k$. For any $\beta \in \mathfrak{B}^t$ and $\mu \in \mathcal{U}^t$, (2.19) shows that

$$E_t \left[\sup_{s \in [t, T]} \left| \tilde{Y}_s^{t, x_1, \mu, \beta(\mu)} \left(T, h(\tilde{X}_T^{t, x_1, \mu, \beta(\mu)}) \right) - \tilde{Y}_s^{t, x_2, \mu, \beta(\mu)} \left(T, h(\tilde{X}_T^{t, x_2, \mu, \beta(\mu)}) \right) \right|^q \right] \leq c_0 |x_1 - x_2|^2.$$

It then follows that

$$\begin{aligned} \tilde{Y}_t^{t, x_2, \mu, \beta(\mu)} \left(T, h(\tilde{X}_T^{t, x_2, \mu, \beta(\mu)}) \right) - c_0 |x_1 - x_2|^{2/q} &\leq \tilde{Y}_t^{t, x_1, \mu, \beta(\mu)} \left(T, h(\tilde{X}_T^{t, x_1, \mu, \beta(\mu)}) \right) \\ &\leq \tilde{Y}_t^{t, x_2, \mu, \beta(\mu)} \left(T, h(\tilde{X}_T^{t, x_2, \mu, \beta(\mu)}) \right) + c_0 |x_1 - x_2|^{2/q}. \end{aligned} \quad (2.28)$$

Taking supremum over $\mu \in \mathcal{U}^t$ and then taking infimum over $\beta \in \mathfrak{B}^t$ yield that

$$w_1(t, x_2) - c_0 |x_1 - x_2|^{2/q} \leq w_1(t, x_1) \leq w_1(t, x_2) + c_0 |x_1 - x_2|^{2/q}.$$

Thus $|w_1(t, x_1) - w_1(t, x_2)| \leq c_0 |x_1 - x_2|^{2/q}$, and one can deduce the similar inequalities for \hat{w}_1 , w_2 and \hat{w}_2 :

Proposition 2.1. For any $t \in [0, T]$ and $x_1, x_2 \in \mathbb{R}^k$, we have

$$|w_1(t, x_1) - w_1(t, x_2)| + |\hat{w}_1(t, x_1) - \hat{w}_1(t, x_2)| + |w_2(t, x_1) - w_2(t, x_2)| + |\hat{w}_2(t, x_1) - \hat{w}_2(t, x_2)| \leq c_0 |x_1 - x_2|^{2/q}.$$

However, these value functions are generally not $1/q$ -Hölder continuous in t unless the control spaces are compact.

Remark 2.2. When trying to directly prove the dynamic programming principle, [18] encountered a measurability issue; see page 299 therein. To overcome this technical difficulty, they first proved that the value functions are unique viscosity solutions to the associated Bellman-Isaacs equations by a time-discretization approach (assuming that the limiting Isaacs equation has a comparison principle), which relies on the following regularity of the approximating values v_π

$$|v_\pi(t, x) - v_\pi(t', x')| \leq C(|t - t'|^{1/2} + |x - x'|) \quad \forall (t, x), (t', x') \in [0, T] \times \mathbb{R}^k$$

with a uniform coefficient $C > 0$ for all partitions π of $[0, T]$. Since our value functions are not $1/2$ -Hölder continuous in t given $q = 2$, this method does not work in general under our assumptions. Instead, we specify Elliott-Kalton strategies as measurable random fields from one control space to another in order to avoid similar measurability issues when pasting strategies (see Proposition 4.10). This is a crucial ingredient in the proof of the supersolution (resp. subsolution) side of the dynamic programming principle (Theorem 2.1) for w_1 (resp. w_2).

Given $i = 1, 2$, since $w_i(t, \cdot)$ is continuous for any $t \in [0, T]$, one can deduce that for any \mathbf{F}^t -stopping time τ with countably many values $\{t_n\}_{n \in \mathbb{N}} \subset [t, T]$, and any \mathbb{R}^k -valued, \mathcal{F}_τ^t -measurable random variable ξ

$$w_i(\tau, \xi) = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{\tau = t_n\}} w_i(t_n, \xi) \text{ is } \mathcal{F}_\tau^t\text{-measurable.} \quad (2.29)$$

Similarly, $\hat{w}_i(\tau, \xi)$ is also \mathcal{F}_τ^t -measurable.

Then we have the following dynamic programming principle for value functions.

Theorem 2.1. *Let $(t, x) \in [0, T] \times \mathbb{R}^k$. For any family $\{\tau_{\mu, \beta} : \mu \in \mathcal{U}^t, \beta \in \mathfrak{B}^t\}$ of $\mathbb{Q}_{t, T}$ -valued, \mathbf{F}^t -stopping times*

$$w_1(t, x) \leq \inf_{\beta \in \mathfrak{B}^t} \sup_{\mu \in \mathcal{U}^t} \tilde{Y}_t^{t, x, \mu, \beta \langle \mu \rangle} \left(\tau_{\mu, \beta}, w_1(\tau_{\mu, \beta}, \tilde{X}_{\tau_{\mu, \beta}}^{t, x, \mu, \beta \langle \mu \rangle}) \right) \quad (2.30)$$

$$\text{and } \hat{w}_1(t, x) \leq \inf_{\beta \in \mathfrak{B}^t} \sup_{\mu \in \mathcal{U}^t} \tilde{Y}_t^{t, x, \mu, \beta \langle \mu \rangle} \left(\tau_{\mu, \beta}, \hat{w}_1(\tau_{\mu, \beta}, \tilde{X}_{\tau_{\mu, \beta}}^{t, x, \mu, \beta \langle \mu \rangle}) \right); \quad (2.31)$$

the reverse inequality (of (2.31)) holds if

$$(\mathbf{V}_\lambda) \quad \begin{cases} \underline{L}, \bar{l} \text{ and } h \text{ satisfy (2.20); } b, \sigma \text{ are } \lambda\text{-H\"older continuous in } v \text{ (see (2.12)); and} \\ f \text{ is } 2\lambda\text{-H\"older continuous in } v \text{ (see (2.23)) for some } \lambda \in (0, 1). \end{cases}$$

On the other hand, for any family $\{\tau_{\nu, \alpha} : \nu \in \mathcal{V}^t, \alpha \in \mathcal{A}^t\}$ of $\mathbb{Q}_{t, T}$ -valued, \mathbf{F}^t -stopping times

$$w_2(t, x) \geq \sup_{\alpha \in \mathcal{A}^t} \inf_{\nu \in \mathcal{V}^t} \tilde{Y}_t^{t, x, \alpha \langle \nu \rangle, \nu} \left(\tau_{\nu, \alpha}, w_2(\tau_{\nu, \alpha}, \tilde{X}_{\tau_{\nu, \alpha}}^{t, x, \alpha \langle \nu \rangle, \nu}) \right) \quad (2.32)$$

$$\text{and } \hat{w}_2(t, x) \geq \sup_{\alpha \in \mathcal{A}^t} \inf_{\nu \in \mathcal{V}^t} \tilde{Y}_t^{t, x, \alpha \langle \nu \rangle, \nu} \left(\tau_{\nu, \alpha}, \hat{w}_2(\tau_{\nu, \alpha}, \tilde{X}_{\tau_{\nu, \alpha}}^{t, x, \alpha \langle \nu \rangle, \nu}) \right); \quad (2.33)$$

the reverse inequality of (2.33) holds if

$$(\mathbf{U}_\lambda) \quad \begin{cases} \underline{L}, \bar{l} \text{ and } h \text{ satisfy (2.20); } b, \sigma \text{ are } \lambda\text{-H\"older continuous in } u \text{ (see (2.10)); and} \\ f \text{ is } 2\lambda\text{-H\"older continuous in } u \text{ (see (2.21)) for some } \lambda \in (0, 1). \end{cases}$$

Note that each $\tilde{Y}_t^{t, x, \mu, \beta \langle \mu \rangle} \left(\tau_{\mu, \beta}, w_1(\tau_{\mu, \beta}, \tilde{X}_{\tau_{\mu, \beta}}^{t, x, \mu, \beta \langle \mu \rangle}) \right)$ in (2.30) is well-posed since $w_1 \left(\tau_{\mu, \beta}, \tilde{X}_{\tau_{\mu, \beta}}^{t, x, \mu, \beta \langle \mu \rangle} \right)$ is $\mathcal{F}_{\tau_{\mu, \beta}}^t$ -measurable by (2.29) and since

$$\underline{L}_{\tau_{\mu, \beta}}^{t, x, \mu, \beta \langle \mu \rangle} = \underline{l} \left(\tau_{\mu, \beta}, \tilde{X}_{\tau_{\mu, \beta}}^{t, x, \mu, \beta \langle \mu \rangle} \right) \leq w_1 \left(\tau_{\mu, \beta}, \tilde{X}_{\tau_{\mu, \beta}}^{t, x, \mu, \beta \langle \mu \rangle} \right) \leq \bar{l} \left(\tau_{\mu, \beta}, \tilde{X}_{\tau_{\mu, \beta}}^{t, x, \mu, \beta \langle \mu \rangle} \right) = \bar{L}_{\tau_{\mu, \beta}}^{t, x, \mu, \beta \langle \mu \rangle}$$

by (2.27). The proof of Theorem 2.1 (see Subsection 6.2) relies on (2.22), (2.24), properties of shifted processes (especially shifted SDEs) as well as stability under pasting of controls/strategies, the latter two of which will be discussed in Section 4.

3 An Obstacle Problem for Fully non-linear PDEs

In this section, we show that the (intrinsic) priority values are (discontinuous) viscosity solutions of the following obstacle problem of a PDE with a fully non-linear Hamiltonian H :

$$\min \left\{ (w - \underline{l})(t, x), \max \left\{ -\frac{\partial}{\partial t} w(t, x) - H(t, x, w(t, x), D_x w(t, x), D_x^2 w(t, x)), (w - \bar{l})(t, x) \right\} \right\} = 0, \quad \forall (t, x) \in (0, T) \times \mathbb{R}^k. \quad (3.1)$$

Definition 3.1. *Let $H : [0, T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{S}_k \rightarrow [-\infty, \infty]$ be an upper (resp. lower) semicontinuous functions with \mathbb{S}_k denoting the set of all $\mathbb{R}^{k \times k}$ -valued symmetric matrices. An upper (resp. lower) semicontinuous function $w : [0, T] \times \mathbb{R}^k \rightarrow \mathbb{R}$ is called a viscosity subsolution (resp. supersolution) of (3.1) if $w(T, x) \leq$ (resp. \geq) $h(x)$, $\forall x \in \mathbb{R}^k$, and if for any $(t_0, x_0, \varphi) \in (0, T) \times \mathbb{R}^k \times \mathbb{C}^{1,2}([0, T] \times \mathbb{R}^k)$ such that $w(t_0, x_0) = \varphi(t_0, x_0)$ and that $w - \varphi$ attains a strict local maximum (resp. strict local minimum) at (t_0, x_0) , we have*

$$\min \left\{ (\varphi - \underline{l})(t_0, x_0), \max \left\{ -\frac{\partial}{\partial t} \varphi(t_0, x_0) - H(t_0, x_0, \varphi(t_0, x_0), D_x \varphi(t_0, x_0), D_x^2 \varphi(t_0, x_0)), (\varphi - \bar{l})(t_0, x_0) \right\} \right\} \leq (\text{resp. } \geq) 0.$$

Although the function H in Definition 3.1 may take $\pm\infty$ values, the left-hand-side of the inequality above is between $(w - \bar{l})(t_0, x_0)$ and $(w - \underline{l})(t_0, x_0)$ and thus finite.

For any $(t, x, y, z, \Gamma, u, v) \in [0, T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_k \times \mathbb{U}_0 \times \mathbb{V}_0$, we set

$$H(t, x, y, z, \Gamma, u, v) \triangleq \frac{1}{2} \text{trace}(\sigma \sigma^T(t, x, u, v) \Gamma) + z \cdot b(t, x, u, v) + f(t, x, y, z \cdot \sigma(t, x, u, v), u, v)$$

and consider the following Hamiltonian functions:

$$\begin{aligned} \underline{H}_1(\Xi) &\triangleq \sup_{u \in \mathbb{U}_0} \lim_{\Xi' \rightarrow \Xi} \inf_{v \in \mathbb{V}_0} H(\Xi', u, v), & \overline{H}_1(\Xi) &\triangleq \lim_{n \rightarrow \infty} \downarrow \sup_{u \in \mathbb{U}_0} \inf_{v \in \mathcal{O}_u^n} \overline{\lim}_{\mathbb{U}_0 \ni u' \rightarrow u} \sup_{\Xi' \in O_{1/n}(\Xi)} H(\Xi', u', v), \\ \text{and } \overline{H}_2(\Xi) &\triangleq \inf_{v \in \mathbb{V}_0} \overline{\lim}_{\Xi' \rightarrow \Xi} \sup_{u \in \mathbb{U}_0} H(\Xi', u, v), & \underline{H}_2(\Xi) &\triangleq \lim_{n \rightarrow \infty} \uparrow \inf_{v \in \mathbb{V}_0} \sup_{u \in \mathcal{O}_v^n} \lim_{\mathbb{U}_0 \ni u' \rightarrow u} \inf_{\Xi' \in O_{1/n}(\Xi)} H(\Xi', u', v), \end{aligned}$$

where $\Xi = (t, x, y, z, \Gamma)$, $\mathcal{O}_u^n \triangleq \{v \in \mathbb{V}_0 : [v]_{\mathbb{V}} \leq n + n[u]_{\mathbb{U}}\}$ and $\mathcal{O}_v^n \triangleq \{u \in \mathbb{U}_0 : [u]_{\mathbb{U}} \leq n + n[v]_{\mathbb{V}}\}$.

For any $(t, x) \in [0, T] \times \mathbb{R}^k$, Proposition 2.1 implies that

$$w_1^*(t, x) \triangleq \overline{\lim}_{t' \rightarrow t} w_1(t', x) = \overline{\lim}_{(t', x') \rightarrow (t, x)} w_1(t', x') \quad \text{and} \quad w_2^*(t, x) \triangleq \lim_{t' \rightarrow t} w_2(t', x) = \lim_{(t', x') \rightarrow (t, x)} w_2(t', x').$$

In fact, w_1^* is the smallest upper semicontinuous function above w_1 (also known as the upper semicontinuous envelope of w_1), while w_2^* is the largest lower semicontinuous function below w_2 (also known as the lower semicontinuous envelope of w_2). Similarly, for $i = 1, 2$,

$$\underline{w}_i(t, x) \triangleq \lim_{t' \rightarrow t} \widehat{w}_i(t', x) \quad \text{and} \quad \overline{w}_i(t, x) \triangleq \overline{\lim}_{t' \rightarrow t} \widehat{w}_i(t', x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^k$$

are the lower and upper semicontinuous envelopes of \widehat{w}_i respectively.

Given $x \in \mathbb{R}^k$, though $w_i(T, x) = \widehat{w}_i(T, x) = h(x)$, probably neither of $w_i^*(x)$, $\underline{w}_i(x)$, $\overline{w}_i(x)$ equals to $h(x)$ as the value functions w_i , \widehat{w}_i may not be continuous in t .

Theorem 3.1.

- 1) If \mathbb{U}_0 (resp. \mathbb{V}_0) is a countable union of closed subsets of \mathbb{U} (resp. \mathbb{V}), then \overline{w}_1 and w_1^* (resp. \underline{w}_2 and w_2^*) are two viscosity subsolutions (resp. supersolutions) of (3.1) with the fully nonlinear Hamiltonian \overline{H}_1 (resp. \underline{H}_2).
- 2) On the other hand, if (\mathbf{V}_λ) (resp. (\mathbf{U}_λ)) holds for some $\lambda \in (0, 1)$, then \underline{w}_1 (resp. \overline{w}_2) is a viscosity supersolution (resp. subsolution) of (3.1) with the fully nonlinear Hamiltonian \underline{H}_1 (resp. \overline{H}_2).

4 Shifted Processes

In this section, we fix $0 \leq t \leq s \leq T$ and explore properties of shifted processes from Ω^t to Ω^s , which are necessary for Section 2 and Section 3.

4.1 Concatenation of Sample Paths

We concatenate an $\omega \in \Omega^t$ and an $\tilde{\omega} \in \Omega^s$ at time s by:

$$(\omega \otimes_s \tilde{\omega})(r) \triangleq \omega(r) \mathbf{1}_{\{r \in [t, s]\}} + (\omega(s) + \tilde{\omega}(r)) \mathbf{1}_{\{r \in [s, T]\}}, \quad \forall r \in [t, T], \quad (4.1)$$

which is still of Ω^t . Clearly, this concatenation is an associative operation: i.e., for any $r \in [s, T]$ and $\widehat{\omega} \in \Omega^r$

$$(\omega \otimes_s \tilde{\omega}) \otimes_r \widehat{\omega} = \omega \otimes_s (\tilde{\omega} \otimes_r \widehat{\omega}).$$

Given $\omega \in \Omega^t$, we set $\omega \otimes_s \emptyset = \emptyset$ and $\omega \otimes_s \tilde{A} \triangleq \{\omega \otimes_s \tilde{\omega} : \tilde{\omega} \in \tilde{A}\}$ for any non-empty $\tilde{A} \subset \Omega^s$. The next result shows that $A \in \mathcal{F}_s^t$ consists of all branches $\omega \otimes_s \Omega^s$ with $\omega \in A$.

Lemma 4.1. *Let $A \in \mathcal{F}_s^t$. If $\omega \in A$, then $\omega \otimes_s \Omega^s \subset A$ (i.e. $A^{s, \omega} = \Omega^s$). Otherwise, if $\omega \notin A$, then $\omega \otimes_s \Omega^s \subset A^c$ (i.e. $A^{s, \omega} = \emptyset$).*

Also, for any $A \subset \Omega^t$ we set $A^{s, \omega} \triangleq \{\tilde{\omega} \in \Omega^s : \omega \otimes_s \tilde{\omega} \in A\}$ as the projection of A on Ω^s along ω . In particular, $\emptyset^{s, \omega} = \emptyset$. For any $A \subset \tilde{A} \subset \Omega^t$ and any collection $\{A_i\}_{i \in \mathcal{I}}$ of subsets of Ω^t , One can deduce that

$$(A^c)^{s, \omega} = \{\tilde{\omega} \in \Omega^s : \omega \otimes_s \tilde{\omega} \in A^c\} = \Omega^s \setminus \{\tilde{\omega} \in \Omega^s : \omega \otimes_s \tilde{\omega} \in A\} = \Omega^s \setminus A^{s, \omega} = (A^{s, \omega})^c, \quad (4.2)$$

$$A^{s, \omega} = \{\tilde{\omega} \in \Omega^s : \omega \otimes_s \tilde{\omega} \in A\} \subset \{\tilde{\omega} \in \Omega^s : \omega \otimes_s \tilde{\omega} \in \tilde{A}\} = \tilde{A}^{s, \omega}, \quad (4.3)$$

$$\text{and } \left(\bigcup_{i \in \mathcal{I}} A_i \right)^{s, \omega} = \left\{ \tilde{\omega} \in \Omega^s : \omega \otimes_s \tilde{\omega} \in \bigcup_{i \in \mathcal{I}} A_i \right\} = \bigcup_{i \in \mathcal{I}} \{\tilde{\omega} \in \Omega^s : \omega \otimes_s \tilde{\omega} \in A_i\} = \bigcup_{i \in \mathcal{I}} A_i^{s, \omega}. \quad (4.4)$$

Lemma 4.2. *Let $\omega \in \Omega^t$. For any open (resp. closed) subset A of Ω^t , $A^{s,\omega}$ is an open (resp. closed) subset of Ω^s . Moreover, given $r \in [s, T]$, we have $A^{s,\omega} \in \mathcal{F}_r^s$ for any $A \in \mathcal{F}_r^t$ and $\omega \otimes_s \tilde{A} \in \mathcal{F}_r^t$ for any $\tilde{A} \in \mathcal{F}_r^s$.*

For any $\mathcal{D} \subset [t, T] \times \Omega^t$, we accordingly set $\mathcal{D}^{s,\omega} \triangleq \{(r, \tilde{\omega}) \in [s, T] \times \Omega^s : (r, \omega \otimes_s \tilde{\omega}) \in \mathcal{D}\}$. Similar to (4.2)-(4.4), for any $\mathcal{D} \subset \tilde{\mathcal{D}} \subset [t, T] \times \Omega^t$ and any collection $\{\mathcal{D}_i\}_{i \in \mathcal{I}}$ of subsets of $[t, T] \times \Omega^t$, one has

$$([t, T] \times \Omega^t) \setminus \mathcal{D}^{s,\omega} = ([s, T] \times \Omega^s) \setminus \mathcal{D}^{s,\omega} = (\mathcal{D}^{s,\omega})^c, \quad \mathcal{D}^{s,\omega} \subset \tilde{\mathcal{D}}^{s,\omega} \quad \text{and} \quad \left(\bigcup_{i \in \mathcal{I}} \mathcal{D}_i \right)^{s,\omega} = \bigcup_{i \in \mathcal{I}} \mathcal{D}_i^{s,\omega}. \quad (4.5)$$

4.2 Measurability of Shifted Processes

For any \mathbb{M} -valued random variable ξ on Ω^t , we define a shifted random variable $\xi^{s,\omega}$ on Ω^s by $\xi^{s,\omega}(\tilde{\omega}) \triangleq \xi(\omega \otimes_s \tilde{\omega})$, $\forall \tilde{\omega} \in \Omega^s$. And for any \mathbb{M} -valued process $X = \{X_r\}_{r \in [t, T]}$, its corresponding shifted process with respect to s and ω consists of $X_r^{s,\omega} = (X_r)^{s,\omega}$, $\forall r \in [s, T]$. In light of Lemma 4.2, shifted random variables and shifted processes “inherit” measurability in the following way:

Proposition 4.1. *If ξ is \mathcal{F}_r^t -measurable for some $r \in [s, T]$, then $\xi^{s,\omega}$ is \mathcal{F}_r^s -measurable. Moreover, for any \mathbb{M} -valued, \mathbf{F}^t -adapted process $\{X_r\}_{r \in [t, T]}$, the shifted process $\{X_r^{s,\omega}\}_{r \in [s, T]}$ is \mathbf{F}^s -adapted.*

Proposition 4.2. *Given $T_0 \in [s, T]$, $\mathcal{D}^{s,\omega} \in \mathcal{B}([s, T_0]) \otimes \mathcal{F}_{T_0}^s$ for any $\mathcal{D} \in \mathcal{B}([t, T_0]) \otimes \mathcal{F}_{T_0}^t$. Consequently, if $\{X_r\}_{r \in [t, T]}$ is an \mathbb{M} -valued, measurable process on $(\Omega^t, \mathcal{F}_T^t)$ (resp. an \mathbb{M} -valued, \mathbf{F}^t -progressively measurable process), then the shifted process $\{X_r^{s,\omega}\}_{r \in [s, T]}$ is a measurable process on $(\Omega^s, \mathcal{F}_T^s)$ (resp. an \mathbf{F}^s -progressively measurable process). Moreover, we have $\mathcal{D}^{s,\omega} \in \mathcal{P}(\mathbf{F}^s)$ for any $\mathcal{D} \in \mathcal{P}(\mathbf{F}^t)$.*

For any $\mathcal{J} \subset [t, T] \times \Omega^t \times \mathbb{M}$, we set $\mathcal{J}^{s,\omega} \triangleq \{(r, \tilde{\omega}, x) \in [s, T] \times \Omega^s \times \mathbb{M} : (r, \omega \otimes_s \tilde{\omega}, x) \in \mathcal{J}\}$.

Corollary 4.1. *For any $\mathcal{J} \in \mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{M})$, $\mathcal{J}^{s,\omega} \in \mathcal{P}(\mathbf{F}^s) \otimes \mathcal{B}(\mathbb{M})$. Let $\tilde{\mathbb{M}}$ be another generic metric space. If a function $f : [t, T] \times \Omega^t \times \mathbb{M} \rightarrow \tilde{\mathbb{M}}$ is $\mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{M})/\mathcal{B}(\tilde{\mathbb{M}})$ -measurable, then the function $f^{s,\omega}(r, \tilde{\omega}, x) \triangleq f(r, \omega \otimes_s \tilde{\omega}, x)$, $\forall (r, \tilde{\omega}, x) \in [s, T] \times \Omega^s \times \mathbb{M}$ is $\mathcal{P}(\mathbf{F}^s) \otimes \mathcal{B}(\mathbb{M})/\mathcal{B}(\tilde{\mathbb{M}})$ -measurable.*

When $s = \tau(\omega)$ for some \mathbf{F}^t -stopping time τ , we shall simplify the above notations by:

$$\omega \otimes_\tau \tilde{\omega} = \omega \otimes_{\tau(\omega)} \tilde{\omega}, \quad A^{\tau,\omega} = A^{\tau(\omega),\omega}, \quad \mathcal{D}^{\tau,\omega} = \mathcal{D}^{\tau(\omega),\omega}, \quad \xi^{\tau,\omega} = \xi^{\tau(\omega),\omega} \quad \text{and} \quad X^{\tau,\omega} = X^{\tau(\omega),\omega}.$$

The following lemma shows that given an \mathbf{F}^t -stopping time τ , an \mathcal{F}_τ^t -measurable random variable only depends on what happens before τ :

Lemma 4.3. *For any \mathbf{F}^t -stopping time τ and $\xi \in \mathcal{F}_\tau^t$, $\xi^{\tau,\omega} \equiv \xi(\omega)$. In particular, $\tau(\omega \otimes_\tau \Omega^{\tau(\omega)}) = \tau(\omega)$.*

4.3 Integrability of Shifted Processes

In this subsection, let τ be an \mathbf{F}^t -stopping time with countably many values. Using the regular conditional probability distribution (see e.g. [45]), we show below that shifted random variables “inherit” integrability property.

Proposition 4.3. *For any $\xi \in \mathbb{L}^1(\mathcal{F}_T^t)$, it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\xi^{\tau,\omega} \in \mathbb{L}^1(\mathcal{F}_T^{\tau(\omega)}, P_0^{\tau(\omega)})$ and*

$$E_{\tau(\omega)}[\xi^{\tau,\omega}] = E_t[\xi | \mathcal{F}_\tau^t](\omega) \in \mathbb{R}, \quad (4.6)$$

where $E_{\tau(\omega)}$ stands for $E_{P_0^{\tau(\omega)}}$. Consequently, for any $p \in [1, \infty)$ and $\xi \in \mathbb{L}^p(\mathcal{F}_T^t)$, it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\xi^{\tau,\omega} \in \mathbb{L}^p(\mathcal{F}_T^{\tau(\omega)}, P_0^{\tau(\omega)})$.

Corollary 4.2. *For any P_0^t -null set \mathcal{N} , it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\mathcal{N}^{\tau,\omega}$ is a $P_0^{\tau(\omega)}$ -null set. Consequently, for any two real-valued random variables ξ_1 and ξ_2 , if $\xi_1 \leq \xi_2$, P_0^t -a.s., then it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\xi_1^{\tau,\omega} \leq \xi_2^{\tau,\omega}$, $P_0^{\tau(\omega)}$ -a.s.*

Next, let us extend Proposition 4.3 to \mathbb{E} -valued measurable processes.

Proposition 4.4. *Let $\{X_r\}_{r \in [t, T]}$ be an \mathbb{E} -valued measurable process on $(\Omega^t, \mathcal{F}_T^t)$ such that $E_t \left[\left(\int_t^T |X_r|^p dr \right)^{\widehat{p}/p} \right] < \infty$ for some $p, \widehat{p} \in [1, \infty)$. It holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\{X_r^{\tau, \omega}\}_{r \in [\tau(\omega), T]}$ is a measurable process on $(\Omega^{\tau(\omega)}, \mathcal{F}_T^{\tau(\omega)})$ with $E_{\tau(\omega)} \left[\left(\int_{\tau(\omega)}^T |X_r^{\tau, \omega}|^p dr \right)^{\widehat{p}/p} \right] < \infty$.*

Corollary 4.3. *Given $p, \widehat{p} \in [1, \infty)$, if $\{X_r\}_{r \in [t, T]} \in \mathbb{H}_{\mathbf{F}^t}^{p, \widehat{p}}([t, T], \mathbb{E})$ (resp. $\mathbb{C}_{\mathbf{F}^t}^p([t, T], \mathbb{E})$), then it holds for P_0^t -a.s. $\omega \in \Omega^t$, $\{X_r^{\tau, \omega}\}_{r \in [\tau(\omega), T]} \in \mathbb{H}_{\mathbf{F}^{\tau(\omega)}}^{p, \widehat{p}}([\tau(\omega), T], \mathbb{E}, P_0^{\tau(\omega)})$ (resp. $\mathbb{C}_{\mathbf{F}^{\tau(\omega)}}^p([\tau(\omega), T], \mathbb{E}, P_0^{\tau(\omega)})$).*

Similar to Corollary 4.2, a shifted $dr \times dP_0^t$ -null set still has zero product measure:

Proposition 4.5. *For any $\mathcal{D} \in \mathcal{B}([t, T]) \otimes \mathcal{F}_T^t$ with $(dr \times dP_0^t)(\mathcal{D} \cap [\tau, T]) = 0$, it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\mathcal{D}^{\tau, \omega} \in \mathcal{B}([\tau(\omega), T]) \otimes \mathcal{F}_T^{\tau(\omega)}$ with $(dr \times dP_0^{\tau(\omega)})(\mathcal{D}^{\tau, \omega}) = 0$.*

The following analyzes the admissibility of controls and strategies when they are shifted.

Proposition 4.6. (1) *For any $\mu \in \mathcal{U}^t$ (resp. $\nu \in \mathcal{V}^t$), it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\mu^{\tau, \omega} = \{\mu_r^{\tau, \omega}\}_{r \in [\tau(\omega), T]} \in \mathcal{U}^{\tau(\omega)}$ (resp. $\nu^{\tau, \omega} = \{\nu_r^{\tau, \omega}\}_{r \in [\tau(\omega), T]} \in \mathcal{V}^{\tau(\omega)}$).*

(2) *For any $\alpha \in \mathcal{A}^t$ (resp. $\alpha \in \widehat{\mathcal{A}}^t$, $\beta \in \mathfrak{B}^t$ and $\beta \in \widehat{\mathfrak{B}}^t$), it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\alpha^{s, \omega} \in \mathcal{A}^s$ (resp. $\alpha^{s, \omega} \in \widehat{\mathcal{A}}^s$, $\beta^{s, \omega} \in \mathfrak{B}^s$ and $\beta^{s, \omega} \in \widehat{\mathfrak{B}}^s$).*

4.4 Shifted Stochastic Differential Equations

In this subsection, we still consider an \mathbf{F}^t -stopping time τ with countably many values.

Fix $x \in \mathbb{R}^k$, $\mu \in \mathcal{U}^t$, $\nu \in \mathcal{V}^t$ and set $\Theta = (t, x, \mu, \nu)$. For P_0^t -a.s. $\omega \in \Omega^t$, Proposition 4.6 (1) shows that $(\mu^{\tau, \omega}, \nu^{\tau, \omega}) \in \mathcal{U}^{\tau(\omega)} \times \mathcal{V}^{\tau(\omega)}$, and thus we know from Section 2 that the following SDE on the probability space $(\Omega^{\tau(\omega)}, \overline{\mathcal{F}}_T^{\tau(\omega)}, P_0^{\tau(\omega)})$:

$$X_s = \widetilde{X}_{\tau(\omega)}^{\Theta}(\omega) + \int_{\tau(\omega)}^s b(r, X_r, \mu_r^{\tau, \omega}, \nu_r^{\tau, \omega}) dr + \int_{\tau(\omega)}^s \sigma(r, X_r, \mu_r^{\tau, \omega}, \nu_r^{\tau, \omega}) dB_r^{\tau(\omega)}, \quad s \in [\tau(\omega), T] \quad (4.7)$$

admits a unique solution $\{X_s^{\Theta, \omega}\}_{s \in [\tau(\omega), T]} \in \mathbb{C}_{\mathbf{F}^{\tau(\omega)}}^2([\tau(\omega), T], \mathbb{R}^k)$ with $\Theta_{\tau}^{\omega} \triangleq (\tau(\omega), \widetilde{X}_{\tau(\omega)}^{\Theta}(\omega), \mu^{\tau, \omega}, \nu^{\tau, \omega})$. As shown below, the $\mathbf{F}^{\tau(\omega)}$ -version of $\{X_s^{\Theta, \omega}\}_{s \in [\tau(\omega), T]}$ is exactly the shifted process $\{(\widetilde{X}_s^{\Theta})^{\tau, \omega}\}_{s \in [\tau(\omega), T]}$.

Proposition 4.7. *It holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\widetilde{X}_s^{\Theta, \omega} = (\widetilde{X}_s^{\Theta})^{\tau, \omega}$, $\forall s \in [\tau(\omega), T]$.*

This result has appeared in [18] for case of compact control spaces (see the paragraph below (1.16) therein) and appeared in Lemma 3.3 of [33] where only one unbounded control is considered. The proof of Proposition 4.7 depends on the following result about the convergence of shifted random variables in probability.

Lemma 4.4. *For any $\{\xi_i\}_{i \in \mathbb{N}} \subset \mathbb{L}^1(\mathcal{F}_T^t)$ that converges to 0 in probability P_0^t , we can find a subsequence $\{\widehat{\xi}_i\}_{i \in \mathbb{N}}$ of it such that for P_0^t -a.s. $\omega \in \Omega^t$, $\{\widehat{\xi}_i^{\tau, \omega}\}_{i \in \mathbb{N}}$ converges to 0 in probability $P_0^{\tau(\omega)}$.*

For any \mathcal{F}_T^t -measurable random variable ξ with $\underline{L}_T^{\Theta} \leq \xi \leq \overline{L}_T^{\Theta}$, P_0^t -a.s., Proposition 4.1, Corollary 4.2 and Proposition 4.7 imply that for P_0^t -a.s. $\omega \in \Omega^t$, $\xi^{\tau, \omega} \in \mathcal{F}_T^{\tau(\omega)}$ and

$$\underline{L}_T^{\Theta, \omega} = \underline{l}(T, \widetilde{X}_T^{\Theta, \omega}) = \underline{l}(T, (\widetilde{X}_T^{\Theta})^{\tau, \omega}) \leq (\underline{L}_T^{\Theta})^{\tau, \omega} \leq \xi^{\tau, \omega} \leq (\overline{L}_T^{\Theta})^{\tau, \omega} \leq \overline{l}(T, (\widetilde{X}_T^{\Theta})^{\tau, \omega}) = \overline{l}(T, \widetilde{X}_T^{\Theta, \omega}) = \overline{L}_T^{\Theta, \omega}.$$

Then Section 2 also shows that for P_0^t -a.s. $\omega \in \Omega^t$, the DRBSDE $(P_0^{\tau(\omega)}, \xi^{\tau, \omega}, f_T^{\Theta, \omega}, \underline{L}_T^{\Theta, \omega}, \overline{L}_T^{\Theta, \omega})$ on the probability space $(\Omega^{\tau(\omega)}, \overline{\mathcal{F}}_T^{\tau(\omega)}, P_0^{\tau(\omega)})$ admits a unique solution $(Y^{\Theta, \omega}(T, \xi^{\tau, \omega}), Z^{\Theta, \omega}(T, \xi^{\tau, \omega}), \underline{K}^{\Theta, \omega}(T, \xi^{\tau, \omega}), \overline{K}^{\Theta, \omega}(T, \xi^{\tau, \omega})) \in \mathbb{G}_{\mathbf{F}^{\tau(\omega)}}^q([\tau(\omega), T])$. Similar to Proposition 4.7, the $\mathbf{F}^{\tau(\omega)}$ -version of $Y^{\Theta, \omega}(T, \xi^{\tau, \omega})$ coincides with the shifted process $\{(\widetilde{Y}^{\Theta}(T, \xi))_s^{\tau, \omega}\}_{s \in [\tau(\omega), T]}$.

Proposition 4.8. *It holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\tilde{Y}_s^{\Theta_\tau^\omega}(T, \xi^{\tau, \omega}) = (\tilde{Y}^\Theta(T, \xi))_s^{\tau, \omega}$, $\forall s \in [\tau(\omega), T]$. In particular, $\tilde{Y}_{\tau(\omega)}^{\Theta_\tau^\omega}(T, \xi^{\tau, \omega}) = (\tilde{Y}_\tau^\Theta(T, \xi))(\omega)$ for P_0^t -a.s. $\omega \in \Omega^t$.*

Proposition 4.8 can also be shown by *Picard* iteration, see (4.15) of [41] for a BSDE version.

4.5 Pasting of Controls and Strategies

We define $\hat{\Pi}_{t,s}(r, \omega) \triangleq (r, \Pi_{t,s}(\omega))$, $\forall (r, \omega) \in [s, T] \times \Omega^t$. Analogous to Lemma 1.2, one has

Lemma 4.5. *Let $r \in [s, T]$. For any $\mathcal{D} \in \mathcal{B}([s, r]) \otimes \mathcal{F}_r^s$, $\hat{\Pi}_{t,s}^{-1}(\mathcal{D}) \in \mathcal{B}([s, r]) \otimes \mathcal{F}_r^t$ and $(dr \times dP_0^t)(\hat{\Pi}_{t,s}^{-1}(\mathcal{D})) = (dr \times dP_0^s)(\mathcal{D})$. Consequently, the mapping $\hat{\Pi}_{t,s} : [s, T] \times \Omega^t \rightarrow [s, T] \times \Omega^s$ is $\mathcal{P}_s(\mathbf{F}^t)/\mathcal{P}(\mathbf{F}^s)$ -measurable, where $\mathcal{P}_s(\mathbf{F}^t) \triangleq \{\mathcal{D} \in \mathcal{P}(\mathbf{F}^t) : \mathcal{D} \subset [s, T] \times \Omega^t\}$ is a σ -field of $[s, T] \times \Omega^t$.*

Now, we are ready to discuss pasting of controls and strategies.

Proposition 4.9. *Let $\mu \in \mathcal{U}^t$ for some $t \in [0, T]$ and let τ be an \mathbf{F}^t -stopping time taking values in a countable subset $\{t_n\}_{n \in \mathbb{N}}$ of $[t, T]$. Given $N \in \mathbb{N}$, let $\{A_i^n\}_{i=1}^{\ell_n} \subset \mathcal{F}_{t_n}^t$ be disjoint subsets of $\{\tau = t_n\}$ for $n = 1, \dots, N$ and set $A_0 \triangleq \Omega^t \setminus \left(\bigcup_{n=1}^N \bigcup_{i=1}^{\ell_n} A_i^n \right)$. Then for any $\{\mu_i^n\}_{i=1}^{\ell_n} \subset \mathcal{U}^{t_n}$, $n = 1, \dots, N$*

$$\hat{\mu}_r(\omega) \triangleq \begin{cases} (\mu_i^n)_r(\Pi_{t,t_n}(\omega)), & \text{if } (r, \omega) \in [\tau, T]_{A_i^n} = [t_n, T] \times A_i^n \text{ for } n = 1, \dots, N \text{ and } i = 1, \dots, \ell_n, \\ \mu_r(\omega), & \text{if } (r, \omega) \in [t, \tau] \cup [\tau, T]_{A_0} \end{cases} \quad (4.8)$$

defines a \mathcal{U}^t -control such that for any $(r, \omega) \in [\tau, T]$

$$\hat{\mu}_r^{\tau, \omega} = \begin{cases} (\mu_i^n)_r, & \text{if } \omega \in A_i^n \text{ for } n = 1, \dots, N \text{ and } i = 1, \dots, \ell_n, \\ \mu_r^{\tau, \omega}, & \text{if } \omega \in A_0. \end{cases}$$

We can paste $\{\nu_i^n\}_{i=1}^{\ell_n} \subset \mathcal{V}^{t_n}$, $n = 1, \dots, N$ to a $\nu \in \mathcal{V}^t$ with respect to $\{A_i^n\}_{i=1}^{\ell_n}$, $n = 1, \dots, N$ in the same manner.

Proposition 4.10. *Let $\alpha \in \mathcal{A}^t$ (resp. $\hat{\mathcal{A}}^t$) for some $t \in [0, T]$ and let τ be an \mathbf{F}^t -stopping time taking values in a countable subset $\{t_n\}_{n \in \mathbb{N}}$ of $\mathbb{Q}_{t,T}$. Given $N \in \mathbb{N}$, let $\{A_i^n\}_{i=1}^{\ell_n} \subset \mathcal{F}_{t_n}^t$ be disjoint subsets of $\{\tau = t_n\}$ for $n = 1, \dots, N$ and set $A_0 \triangleq \Omega^t \setminus \left(\bigcup_{n=1}^N \bigcup_{i=1}^{\ell_n} A_i^n \right)$. Then for any $\{\alpha_i^n\}_{i=1}^{\ell_n} \subset \mathcal{A}^{t_n}$ (resp. $\hat{\mathcal{A}}^{t_n}$), $n = 1, \dots, N$*

$$\hat{\alpha}(r, \omega, v) \triangleq \begin{cases} \alpha_i^n(r, \Pi_{t,t_n}(\omega), v), & \text{if } (r, \omega) \in [\tau, T]_{A_i^n} = [t_n, T] \times A_i^n \text{ for } n = 1, \dots, N \text{ and } i = 1, \dots, \ell_n, \\ \alpha(r, \omega, v), & \text{if } (r, \omega) \in [t, \tau] \cup [\tau, T]_{A_0}, \end{cases} \quad \forall v \in \mathbb{V} \quad (4.9)$$

is an \mathcal{A}^t -strategy (resp. $\hat{\mathcal{A}}^t$ -strategy) such that given $\nu \in \mathcal{V}^t$, it holds for P_0^t -a.s. $\omega \in \Omega^t$ that for any $r \in [\tau(\omega), T]$

$$(\hat{\alpha}(\nu))_r^{\tau, \omega} = \begin{cases} (\alpha_i^n(\nu^{t_n, \omega}))_r, & \text{if } \omega \in A_i^n \text{ for } n = 1, \dots, N \text{ and } i = 1, \dots, \ell_n, \\ (\alpha(\nu))_r^{\tau, \omega}, & \text{if } \omega \in A_0. \end{cases} \quad (4.10)$$

We can paste $\{\beta_i^n\}_{i=1}^{\ell_n} \subset \mathcal{B}^{t_n}$ (resp. $\hat{\mathcal{B}}^{t_n}$), $n = 1, \dots, N$ to a $\beta \in \mathcal{B}^t$ (resp. $\hat{\mathcal{B}}^t$) with respect to $\{A_i^n\}_{i=1}^{\ell_n}$, $n = 1, \dots, N$ in the same manner.

5 Optimization Problems with Square-Integrable Controls

In this section, we will remove v -controls (or take $\mathbb{V} = \mathbb{V}_0 = \{v_0\}$) so that the zero-sum stochastic differential game discussed above degenerates as an one-control optimization problem for player I.

5.1 General Results

We will follow the setting of Section 2 except that we take away the v -controls from all notations and definitions. In particular, \mathcal{V}^t , \mathfrak{B}^t and $\tilde{\mathfrak{B}}^t$ disappear (or become singletons) while \mathcal{A}^t is equivalent to \mathcal{U}^t . Then for any $(t, x) \in [0, T] \times \mathbb{R}^k$, $w_1(t, x)$, $\hat{w}_1(t, x)$, $w_2(t, x)$ coincide as

$$w(t, x) \triangleq \sup_{\mu \in \mathcal{U}^t} \tilde{Y}_t^{t, x, \mu} \left(T, h(\tilde{X}_T^{t, x, \mu}) \right). \quad (5.1)$$

As (\mathbf{V}_λ) trivially holds for any $\lambda \in (0, 1)$, the one-control version of Theorem 2.1 reads as:

Proposition 5.1. *Let $(t, x) \in [0, T] \times \mathbb{R}^k$. For any family $\{\tau_\mu : \mu \in \mathcal{U}^t\}$ of $\mathbb{Q}_{t, T}$ -valued, \mathbf{F}^t -stopping times*

$$w(t, x) = \sup_{\mu \in \mathcal{U}^t} \tilde{Y}_t^{t, x, \mu} \left(\tau_\mu, w(\tau_\mu, \tilde{X}_{\tau_\mu}^{t, x, \mu}) \right). \quad (5.2)$$

Moreover, for $\Xi = (t, x, y, z, \Gamma) \in [0, T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_k$, $\underline{H}(\Xi)$ and $\overline{H}(\Xi)$ simplify respectively as $\underline{H}(\Xi) \triangleq \sup_{u \in \mathbb{U}_0} \lim_{\Xi' \rightarrow \Xi} H(\Xi', u)$ and

$$\begin{aligned} \overline{H}(\Xi) &= \lim_{n \rightarrow \infty} \downarrow \sup_{u \in \mathbb{U}_0} \overline{\lim}_{\mathbb{U}_0 \ni u' \rightarrow u} \sup_{\Xi' \in O_{1/n}(\Xi)} H(\Xi', u') = \lim_{n \rightarrow \infty} \downarrow \sup_{u \in \mathbb{U}_0} \sup_{\Xi' \in O_{1/n}(\Xi)} H(\Xi', u) \\ &= \lim_{n \rightarrow \infty} \downarrow \sup_{\Xi' \in O_{1/n}(\Xi)} \sup_{u \in \mathbb{U}_0} H(\Xi', u) = \overline{\lim}_{\Xi' \rightarrow \Xi} \sup_{u \in \mathbb{U}_0} H(\Xi', u), \end{aligned}$$

where we used the fact that $\sup_{u \in \mathbb{U}_0} \overline{\lim}_{\mathbb{U}_0 \ni u' \rightarrow u} = \sup_{u \in \mathbb{U}_0}$ in the second equality. Then we have the following one-control version of Theorem 3.1:

Proposition 5.2. *The lower semicontinuous envelopes of w : $\underline{w}(t, x) \triangleq \lim_{t' \rightarrow t} w(t', x)$, $(t, x) \in [0, T] \times \mathbb{R}^k$ is a viscosity supersolution of (3.1) with the fully nonlinear Hamiltonian \underline{H} . On the other hand, if \mathbb{U}_0 is a countable union of closed subsets of \mathbb{U} , then the upper semicontinuous envelopes of w : $\overline{w}(t, x) \triangleq \overline{\lim}_{t' \rightarrow t} w(t', x)$, $(t, x) \in [0, T] \times \mathbb{R}^k$ is a viscosity subsolutions of (3.1) with the fully nonlinear Hamiltonian \overline{H} .*

Remark 5.1. *Similar to [44], we only need to assume the measurability (resp. lower semi-continuity) of the terminal function h for the “ \leq ” (resp. “ \geq ”) inequality of (5.2) and thus for the viscosity subsolution (resp. supersolution) part of Proposition 5.2.*

5.2 Connection to the *Second-Order* Doubly Reflected BSDEs

Now let us further take $k = d$, $\mathbb{U} = \mathbb{S}_d$, $u_0 = 0$ and $\mathbb{U}_0 = \mathbb{S}_d^{>0} \triangleq \{\Gamma \in \mathbb{S}_d : \det(\Gamma) > 0\}$.

Lemma 5.1. *\mathbb{S}_d is a separable normed vector space on which the determinant $\det(\cdot)$ is continuous.*

It follows that $\mathbb{U}_0 = \mathbb{S}_d^{>0}$ consists of closed subsets of \mathbb{U} : $F_n \triangleq \{u \in \mathbb{U} : \det(u) \geq 1/n\}$, $n \in \mathbb{N}$. We also specify:

$$b(t, x, u) = b(t, x) \quad \text{and} \quad \sigma(t, x, u) = u, \quad \forall (t, x, u) \in [0, T] \times \mathbb{R}^d \times \mathbb{U}. \quad (5.3)$$

for a function $b(t, x) : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ that is $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d) / \mathcal{B}(\mathbb{R}^d)$ -measurable and Lipschitz continuous in x with coefficient $\gamma > 0$. Via the transformation (5.12), we will show that the value function w defined in (5.1) is the value function of second-order doubly reflected BSDEs.

Given $t \in [0, T]$, we say that a $P \in \mathcal{P}^t$ is a *semi-martingale measure* if B^t is a continuous semi-martingale with respect to (\mathbf{F}^t, P) . Let \mathcal{Q}^t be the collection of all semi-martingale measures on $(\Omega^t, \mathcal{F}_T^t)$.

Lemma 5.2. *Let $t \in [0, T]$. For $i, j \in \{1, \dots, d\}$, there exists an $\mathbb{R} \cup \{\infty\}$ -valued, \mathbf{F}^t -progressively measurable process $\hat{a}^{t, i, j}$ such that for any $P \in \mathcal{Q}^t$, it holds P -a.s. that*

$$\hat{a}_s^{t, i, j} = \hat{a}_s^{t, j, i} = \overline{\lim}_{m \rightarrow \infty} m \left(\langle B^{t, i}, B^{t, j} \rangle_s^P - \langle B^{t, i}, B^{t, j} \rangle_{(s-1/m)^+}^P \right), \quad s \in [t, T], \quad (5.4)$$

where $\langle B^{t, i}, B^{t, j} \rangle^P$'s denote the P -cross variance between the i -th and j -th components of B^t .

Similar to [43], we let \mathcal{Q}_W^t collect all $P \in \mathcal{Q}^t$ such that P -a.s.

$$\langle B^t \rangle_s^P \text{ is absolutely continuous in } s \text{ and } \hat{a}_s^t \in \mathbb{S}_d^{>0} \text{ for a.e. } s \in [t, T]. \quad (5.5)$$

In general, two different probabilities P_1, P_2 of \mathcal{Q}_W^t are mutually singular, see Example 2.1 of [43].

Lemma 5.3. *For any $t \in [0, T]$, there exist a unique $\mathbb{S}_d^{>0}$ -valued, \mathbf{F}^t -progressively measurable process \hat{q}^t such that for any $P \in \mathcal{Q}_W^t$, it holds P -a.s. that $(\hat{q}_s^t)^2 = \hat{q}_s^t \cdot \hat{q}_s^t = \hat{a}_s^t$ for a.e. $s \in [t, T]$.*

For any $P \in \mathcal{Q}_W^t$, we define $\mathfrak{I}_s^P \triangleq \int_{[t, s]}^P (\hat{q}_r^t)^{-1} dB_r^t, s \in [t, T]$, which is a continuous semi-martingale with respect to (\mathbf{F}^P, P) . Since the first part of (5.5) and (5.4) imply that P -a.s.,

$$\langle B^t \rangle_s^P = \int_t^s \hat{a}_r^t dr, \quad \forall s \in [t, T], \quad (5.6)$$

one can deduce from Lemma 5.3 that P -a.s.

$$\langle \mathfrak{I}^P \rangle_s^P = \int_t^s (\hat{q}_r^t)^{-1} \cdot (\hat{q}_r^t)^{-1} d\langle B^t \rangle_r^P = \int_t^s (\hat{q}_r^t)^{-1} \cdot (\hat{q}_r^t)^{-1} \cdot \hat{a}_r^t dr = (s - t) I_{d \times d}, \quad \forall s \in [t, T].$$

In light of Lévy's characterization, the martingale part W^P of \mathfrak{I}^P is a Brownian motion under P . Let $\mathbf{G}^P = \{\mathcal{G}_s^P\}_{s \in [t, T]}$ denote the P -augmented filtration generated by the P -Brownian motion W^P , i.e.

$$\mathcal{G}_s^P \triangleq \sigma\left(\sigma(W_r^P, r \in [t, s]) \cup \mathcal{N}^P\right), \quad \forall s \in [t, T].$$

Let $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\mu \in \mathcal{U}^t$. According to our specification (5.3), $\{X_s^{t, x, \mu}\}_{s \in [t, T]} \in \mathbb{C}_{\mathbf{F}^t}^2([t, T], \mathbb{R}^k)$ stands for the unique solution of the following SDE on the probability space $(\Omega^t, \mathcal{F}_T^t, P_0^t)$:

$$X_s = x + \int_t^s b(r, X_r) dr + \int_t^s \mu_r dB_r^t, \quad s \in [t, T]. \quad (5.7)$$

Similar to (2.7), we have

$$E_t \left[\sup_{s \in [t, T]} |X_s^{t, x, \mu}|^2 \right] \leq c_0 \left(1 + |x|^2 + E_t \int_t^T |\mu_s|^2 ds \right) < \infty. \quad (5.8)$$

By Lemma 1.3 (2), $X^{t, x, \mu}$ admits a unique \mathbf{F}^t -version $\tilde{X}^{t, x, \mu} \in \mathbb{C}_{\mathbf{F}^t}^2([t, T], \mathbb{R}^k)$ which also satisfies (5.8). As $\tilde{X}^{t, x, \mu}$ has continuous paths except on some $\mathcal{N}_\mu^{t, x} \in \mathcal{F}_T^t$ with $P_0^t(\mathcal{N}_\mu^{t, x}) = 0$, we can view

$$\mathcal{X}^{t, x, \mu} \triangleq \mathbf{1}_{(\mathcal{N}_\mu^{t, x})^c} \left(\tilde{X}^{t, x, \mu} - x \right) \quad (5.9)$$

as a mapping from Ω^t to Ω^t . We claim that $\mathcal{X}^{t, x, \mu}$ is actually a measurable mapping from $(\Omega^t, \mathcal{F}_T^t)$ to $(\Omega^t, \mathcal{F}_T^t)$: To see this, we pick up an arbitrary pair $(s, \mathcal{E}) \in [t, T] \times \mathcal{B}(\mathbb{R}^d)$. The \mathbf{F}^t -adaptedness of $\tilde{X}^{t, x, \mu}$ implies that

$$\begin{aligned} (\mathcal{X}^{t, x, \mu})^{-1} \left((B_s^t)^{-1}(\mathcal{E}) \right) &= \{ \omega \in \Omega^t : \mathcal{X}^{t, x, \mu}(\omega) \in (B_s^t)^{-1}(\mathcal{E}) \} = \{ \omega \in \Omega^t : \mathcal{X}_s^{t, x, \mu}(\omega) \in \mathcal{E} \} \\ &= \begin{cases} \mathcal{N}_\mu^{t, x} \cup \left((\mathcal{N}_\mu^{t, x})^c \cap \{ \omega \in \Omega^t : \tilde{X}_s^{t, x, \mu}(\omega) \in \mathcal{E}_x \} \right) \in \mathcal{F}_T^t, & \text{if } 0 \in \mathcal{E}, \\ (\mathcal{N}_\mu^{t, x})^c \cap \{ \omega \in \Omega^t : \tilde{X}_s^{t, x, \mu}(\omega) \in \mathcal{E}_x \} \in \mathcal{F}_T^t, & \text{if } 0 \notin \mathcal{E}, \end{cases} \end{aligned} \quad (5.10)$$

where $\mathcal{E}_x = \{x + x' : x' \in \mathcal{E}\} \in \mathcal{B}(\mathbb{R}^d)$. Thus $(B_s^t)^{-1}(\mathcal{E}) \in \Lambda^t \triangleq \{A \subset \Omega^t : (\mathcal{X}^{t, x, \mu})^{-1}(A) \in \mathcal{F}_T^t\}$. Clearly, Λ^t is a σ -field of Ω^t . It follows that

$$\mathcal{F}_T^t = \sigma \left((B_s^t)^{-1}(\mathcal{E}); s \in [t, T], \mathcal{E} \in \mathcal{B}(\mathbb{R}^d) \right) \subset \Lambda^t, \quad (5.11)$$

proving the measurability of the mapping $\mathcal{X}^{t, x, \mu}$. Consequently, we can induce a probability measure

$$P^{t, x, \mu} \triangleq P_0^t \circ (\mathcal{X}^{t, x, \mu})^{-1} \quad (5.12)$$

on $(\Omega^t, \mathcal{F}_T^t)$, i.e. $P^{t, x, \mu} \in \mathcal{P}^t$. Similar to [43], we set $\mathcal{Q}_S^{t, x} \triangleq \{P^{t, x, \mu}\}_{\mu \in \mathcal{U}^t}$.

Lemma 5.4. *Given $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\mu \in \mathcal{U}^t$, let $\mathcal{X}^{t,x,\mu} : \Omega^t \rightarrow \Omega^t$ be the mapping defined in (5.9). It holds for any $s \in [t, T]$ that $(\mathcal{X}^{t,x,\mu})^{-1}(\mathcal{F}_s^{P^{t,x,\mu}}) \subset \overline{\mathcal{F}}_s^t$. Moreover, we have*

$$P^{t,x,\mu} = P_0^t \circ (\mathcal{X}^{t,x,\mu})^{-1} \quad \text{on } \mathcal{F}_T^{P^{t,x,\mu}}. \quad (5.13)$$

Proposition 5.3. *For any $(t, x) \in [0, T] \times \mathbb{R}^d$, we have $\mathcal{Q}_S^{t,x} \subset \mathcal{Q}_W^t$.*

The following result about $\mathcal{Q}_S^{t,x}$ is inspired by Lemma 8.1 of [43].

Proposition 5.4. *Let $(t, x) \in [0, T] \times \mathbb{R}^d$. For any $P \in \mathcal{Q}_S^{t,x}$, \mathbf{F}^P coincides with \mathbf{G}^P , the P -augmented filtration generated by the P -Brownian motion W^P .*

Fix $(t, x) \in [0, T] \times \mathbb{R}^d$ and set $B^{t,x} \triangleq x + B^t$. By the continuity of \underline{l} and \bar{l} , $\underline{\mathcal{L}}_s^{t,x} \triangleq \underline{l}(s, B_s^{t,x})$ and $\overline{\mathcal{L}}_s^{t,x} \triangleq \bar{l}(s, B_s^{t,x})$, $s \in [t, T]$ are two real-valued, \mathbf{F}^t -adapted continuous processes satisfying $\underline{\mathcal{L}}_s^{t,x} < \overline{\mathcal{L}}_s^{t,x}$, $\forall s \in [t, T]$. The measurability of $(f, B^{t,x})$, the measurability of \hat{q}^t by Lemma (5.3) as well as the Lipschitz continuity of f in (y, z) imply that

$$\hat{f}^{t,x}(s, \omega, y, z) \triangleq f(s, B_s^{t,x}(\omega), y, z, \hat{q}_s^t(\omega)), \quad \forall (s, \omega, y, z) \in [t, T] \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d$$

is a $\mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$ -measurable function that is also Lipschitz continuous in (y, z) .

Given $\mu \in \mathcal{U}^t$, one can deduce from (2.3), the version of (2.4) and (2.5) without ν -controls, Hölder's inequality, (6.144) and (5.8) that

$$\begin{aligned} E_{P^{t,x,\mu}} \left[\sup_{s \in [t, T]} |\underline{\mathcal{L}}_s^{t,x}|^q + \sup_{s \in [t, T]} |\overline{\mathcal{L}}_s^{t,x}|^q + \left(\int_t^T |\hat{f}_\tau^{t,x}(s, 0, 0)| ds \right)^q \right] &\leq c_0 + c_0 E_{P^{t,x,\mu}} \left[\sup_{s \in [t, T]} |B_s^{t,x}|^2 + \int_t^T |\hat{q}_s^t|^2 ds \right] \\ &\leq c_0 + c_0 E_t \left[\sup_{s \in [t, T]} |B_s^{t,x}(\mathcal{X}^{t,x,\mu})|^2 + \int_t^T |\hat{q}_s^t(\mathcal{X}^{t,x,\mu})|^2 ds \right] \leq c_0 + c_0 E_t \left[\sup_{s \in [t, T]} |\tilde{X}_s^{t,x,\mu}|^2 + \int_t^T |\mu_s|^2 ds \right] < \infty. \end{aligned}$$

As $\underline{\mathcal{L}}_T^{t,x} \leq h(B_T^{t,x}) \leq \overline{\mathcal{L}}_T^{t,x}$, it follows that $E_{P^{t,x,\mu}}[|h(B_T^{t,x})|^q] < \infty$, i.e. $\xi \in \mathbb{L}^q(\mathcal{F}_T^t, P^{t,x,\mu})$. Then Proposition 5.4 and Theorem 4.1 of [15] shows that the following Doubly reflected BSDE on the probability space $(\Omega^t, \mathcal{F}_T^{P^{t,x,\mu}}, P^{t,x,\mu})$

$$\begin{cases} \mathcal{Y}_s = h(B_T^{t,x}) + \int_s^T \hat{f}^{t,x}(r, \mathcal{Y}_r, \mathcal{Z}_r) dr + \underline{\mathcal{K}}_T - \underline{\mathcal{K}}_s - (\overline{\mathcal{K}}_T - \overline{\mathcal{K}}_s) - \int_{[s, T]} \mathcal{Z}_r dW_r^{P^{t,x,\mu}}, & s \in [t, T], \\ \underline{\mathcal{L}}_s^{t,x} \leq \mathcal{Y}_s \leq \overline{\mathcal{L}}_s^{t,x}, & s \in [t, T], \quad \text{and} \quad \int_t^T (\mathcal{Y}_s - \underline{\mathcal{L}}_s^{t,x}) d\underline{\mathcal{K}}_s = \int_t^T (\overline{\mathcal{L}}_s^{t,x} - \mathcal{Y}_s) d\overline{\mathcal{K}}_s = 0 \end{cases}$$

admits a unique solution $(\mathcal{Y}^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x})), \mathcal{Z}^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x})), \underline{\mathcal{K}}^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x})), \overline{\mathcal{K}}^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x}))) \in \mathbb{G}_{\mathbf{F}^{P^{t,x,\mu}}}^q([t, T], P^{t,x,\mu})$.

Proposition 5.5. *For any $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\mu \in \mathcal{U}^t$*

$$P_0^t \left(\left(\mathcal{Y}_s^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x})) \right) (\mathcal{X}^{t,x,\mu}) = Y_s^{t,x,\mu}(T, h(\tilde{X}_T^{t,x,\mu})), \quad \forall s \in [t, T] \right) = 1.$$

Let $\tilde{\mathcal{Y}}_s^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x}))$ be the \mathbf{F}^t -version of $\mathcal{Y}_s^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x}))$. For the constant $y^{t,x,\mu} \triangleq \tilde{Y}_t^{t,x,\mu}(T, h(\tilde{X}_T^{t,x,\mu})) \in \mathcal{F}_t^t$, one can deduce from (5.13) that

$$1 = P_0^t \left\{ \left(\mathcal{Y}_t^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x})) \right) (\mathcal{X}^{t,x,\mu}) = y^{t,x,\mu} \right\} = P^{t,x,\mu} \left\{ \mathcal{Y}_t^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x})) = y^{t,x,\mu} \right\}.$$

It follows that $y^{t,x,\mu} = \tilde{\mathcal{Y}}_t^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x}))$. Hence,

$$w(t, x) = \sup_{\mu \in \mathcal{U}^t} \tilde{Y}_t^{t,x,\mu}(T, h(\tilde{X}_T^{t,x,\mu})) = \sup_{P \in \mathcal{Q}_S^{t,x}} \tilde{\mathcal{Y}}_t^{t,x,P}(T, h(B_T^{t,x})), \quad (5.14)$$

which extended the value function of [44] (see (5.9) therein) to the case of doubly reflected BSDEs based on more general forward SDEs. Thus our value function w is closely related to the second-order doubly reflected BSDEs. On the other hand, when the generator $f \equiv 0$, the right-hand-side of (5.14) is a doubly reflected version of the value function considered in [33].

6 Proofs

6.1 Proofs of Section 1 & 2

Proof of Lemma 1.1: For any $s \in [t, T]$, it is clear that $\sigma(\mathcal{C}_s^{t,T}) \subset \sigma\left\{(B_r^{t,T})^{-1}(\mathcal{E}) : r \in [t, s], \mathcal{E} \in \mathcal{B}(\mathbb{R}^d)\right\} = \mathcal{F}_s^{t,T}$. To see the reverse, we fix $r \in [t, s]$. For any $x \in \mathbb{Q}^d$ and $\lambda \in \mathbb{Q}_+$, let $\{s_j\}_{j \in \mathbb{N}} \subset \mathbb{Q}_{r,s}$ with $\lim_{j \rightarrow \infty} s_j = r$. Since $\Omega^{t,T}$ is the set of \mathbb{R}^d -valued continuous functions on $[t, T]$ starting from 0, we can deduce that

$$(B_r^{t,T})^{-1}(O_\lambda(x)) = \bigcup_{n=\lceil \frac{2}{\lambda} \rceil}^{\infty} \bigcup_{m \in \mathbb{N}} \bigcap_{j \geq m} \left((B_{s_j}^{t,T})^{-1}(O_{\lambda - \frac{1}{n}}(x)) \right) \in \sigma(\mathcal{C}_s^{t,T}).$$

which implies that

$$\mathcal{O} \triangleq \{O_\lambda(x) : x \in \mathbb{Q}^d, \lambda \in \mathbb{Q}_+\} \subset \Lambda_r \triangleq \left\{ \mathcal{E} \subset \mathbb{R}^d : (B_r^{t,T})^{-1}(\mathcal{E}) \in \sigma(\mathcal{C}_s^{t,T}) \right\}.$$

Clearly, \mathcal{O} generates $\mathcal{B}(\mathbb{R}^d)$ and Λ_r is a σ -field of \mathbb{R}^d . Thus, one has $\mathcal{B}(\mathbb{R}^d) \subset \Lambda_r$. Then it follows that

$$\mathcal{F}_s^{t,T} = \sigma\left\{(B_r^{t,T})^{-1}(\mathcal{E}) : r \in [t, s], \mathcal{E} \in \mathcal{B}(\mathbb{R}^d)\right\} \subset \sigma(\mathcal{C}_s^{t,T}). \quad \square$$

Proof of Lemma 1.2: For simplicity, let us denote $\Pi_{t,s}^{T,S}$ by Π . We first show the continuity of Π . Let A be an open subset of $\Omega^{s,S}$. Given $\omega \in \Pi^{-1}(A)$, since $\Pi(\omega) \in A$, there exist a $\delta > 0$ such that $O_\delta(\Pi(\omega)) \triangleq \left\{ \tilde{\omega} \in \Omega^{s,S} : \sup_{r \in [s, S]} |\tilde{\omega}(r) - (\Pi(\omega))(r)| < \delta \right\} \subset A$. For any $\omega' \in O_{\delta/2}(\omega)$, one can deduce that

$$\sup_{r \in [s, S]} |(\Pi(\omega'))(r) - (\Pi(\omega))(r)| \leq |\omega'(s) - \omega(s)| + \sup_{r \in [s, S]} |\omega'(r) - \omega(r)| \leq 2 \sup_{r \in [t, T]} |\omega'(r) - \omega(r)| < \delta,$$

which shows that $\Pi(\omega') \in O_\delta(\Pi(\omega)) \subset A$ or $\omega' \in \Pi^{-1}(A)$. Hence, $\Pi^{-1}(A)$ is an open subset of $\Omega^{t,T}$.

Now, let $r \in [s, S]$. For any $s' \in [s, r]$ and $\mathcal{E} \in \mathcal{B}(\mathbb{R}^d)$, one can deduce that

$$\Pi^{-1}\left((B_{s'}^{s,S})^{-1}(\mathcal{E})\right) = \left\{ \omega \in \Omega^{t,T} : B_{s'}^{s,S}(\Pi(\omega)) \in \mathcal{E} \right\} = \left\{ \omega \in \Omega^{t,T} : \omega(s') - \omega(s) \in \mathcal{E} \right\} = (B_{s'}^{t,T} - B_s^{t,T})^{-1}(\mathcal{E}) \in \mathcal{F}_r^{t,T}. \quad (6.1)$$

Thus all the generating sets of $\mathcal{F}_r^{s,S}$ belong to $\Lambda \triangleq \{A \subset \Omega^{s,S} : \Pi^{-1}(A) \in \mathcal{F}_r^{t,T}\}$, which is clearly a σ -field of $\Omega^{s,S}$. It follows that $\mathcal{F}_r^{s,S} \subset \Lambda$, i.e., $\Pi^{-1}(A) \in \mathcal{F}_r^{t,T}$ for any $A \in \mathcal{F}_r^{s,S}$.

Next, we show that the induced probability $\tilde{P} \triangleq P_0^{t,T} \circ \Pi^{-1}$ equals to $P_0^{s,S}$ on $\mathcal{F}_S^{s,S}$: Since the Wiener measure on $(\Omega^{s,S}, \mathcal{B}(\Omega^{s,S}))$ is unique (see e.g. Proposition I.3.3 of [39]), it suffices to show that the canonical process $B^{s,S}$ is a Brownian motion on $\Omega^{s,S}$ under \tilde{P} : Let $s \leq r \leq r' \leq S$. For any $\mathcal{E} \in \mathcal{B}(\mathbb{R}^d)$, similar to (6.1), one can deduce that

$$\Pi^{-1}\left((B_{r'}^{s,S} - B_r^{s,S})^{-1}(\mathcal{E})\right) = (B_{r'}^{t,T} - B_r^{t,T})^{-1}(\mathcal{E}). \quad (6.2)$$

Thus, $\tilde{P}\left((B_{r'}^{s,S} - B_r^{s,S})^{-1}(\mathcal{E})\right) = P_0^{t,T}\left(\Pi^{-1}\left((B_{r'}^{s,S} - B_r^{s,S})^{-1}(\mathcal{E})\right)\right) = P_0^{t,T}\left((B_{r'}^{t,T} - B_r^{t,T})^{-1}(\mathcal{E})\right)$, which shows that the distribution of $B_{r'}^{s,S} - B_r^{s,S}$ under \tilde{P} is the same as that of $B_{r'}^{t,T} - B_r^{t,T}$ under $P_0^{t,T}$ (a d -dimensional normal distribution with mean 0 and variance matrix $(r' - r)I_{d \times d}$).

On the other hand, for any $A \in \mathcal{F}_r^{s,S}$, since $\Pi^{-1}(A)$ belongs to $\mathcal{F}_r^{t,T}$, its independence from $B_{r'}^{t,T} - B_r^{t,T}$ under $P_0^{t,T}$ and (6.2) yield that for any $\mathcal{E} \in \mathcal{B}(\mathbb{R}^d)$

$$\begin{aligned} \tilde{P}\left(A \cap (B_{r'}^{s,S} - B_r^{s,S})^{-1}(\mathcal{E})\right) &= P_0^{t,T}\left(\Pi^{-1}(A) \cap \Pi^{-1}\left((B_{r'}^{s,S} - B_r^{s,S})^{-1}(\mathcal{E})\right)\right) \\ &= P_0^{t,T}\left(\Pi^{-1}(A)\right) \cdot P_0^{t,T}\left(\Pi^{-1}\left((B_{r'}^{s,S} - B_r^{s,S})^{-1}(\mathcal{E})\right)\right) = \tilde{P}(A) \cdot \tilde{P}\left((B_{r'}^{s,S} - B_r^{s,S})^{-1}(\mathcal{E})\right). \end{aligned}$$

Hence, $B_{r'}^{s,S} - B_r^{s,S}$ is independent of $\mathcal{F}_r^{s,S}$ under \tilde{P} . \square

Proof of Lemma 1.3:

(1) First, let $\xi \in \mathbb{L}^1(\mathcal{F}_T^P, P)$ and $s \in [t, T]$. For any $A \in \mathcal{F}_s^P$, there exists an $\tilde{A} \in \mathcal{F}_s^t$ such that $A \Delta \tilde{A} \in \mathcal{N}^P$ (see e.g. Problem 2.7.3 of [27]). Thus we have that $\int_A \xi dP = \int_{\tilde{A}} \xi dP = \int_{\tilde{A}} E_P[\xi | \mathcal{F}_s^t] dP = \int_A E_P[\xi | \mathcal{F}_s^t] dP$, which implies that

$$E_P[\xi | \mathcal{F}_s^P] = E_P[\xi | \mathcal{F}_s^t], \quad P\text{-a.s.} \quad (6.3)$$

Then it easily follows that any martingale X with respect to (\mathbf{F}^t, P) is also a martingale with respect to (\mathbf{F}^P, P) .

Next, let $X = \{X_s\}_{s \in [t, T]}$ be a local martingale with respect to (\mathbf{F}^t, P) . There exists an increasing sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of \mathbf{F}^t -stopping times with $P\left(\lim_{n \rightarrow \infty} \uparrow \tau_n = T\right) = 1$ such that $B_{\tau_n \wedge \cdot}^t, n \in \mathbb{N}$ are all martingales with respect to (\mathbf{F}^t, P) . For any $m \in \mathbb{N}$, $\nu_m \triangleq \inf\{s \in [t, T] : |X_s| > m\} \wedge T$ defines an \mathbf{F}^t -stopping time. In light of the Optional Sampling theorem, $X_{\tau_n \wedge \nu_m \wedge \cdot}$ is a martingale with respect to (\mathbf{F}^t, P) . Thus, for any $t \leq s < r \leq T$, one has

$$E_P[X_{\tau_n \wedge \nu_m \wedge r} | \mathcal{F}_s^P] = E_P[X_{\tau_n \wedge \nu_m \wedge r} | \mathcal{F}_s^t] = X_{\tau_n \wedge \nu_m \wedge s}, \quad P\text{-a.s.} \quad (6.4)$$

Since $P\left(\lim_{n \rightarrow \infty} \uparrow \tau_n = T\right) = 1$, when $n \rightarrow \infty$ in (6.4), the bounded convergence theorem implies that $E_P[X_{\nu_m \wedge r} | \mathcal{F}_s^P] = X_{\nu_m \wedge s}, P\text{-a.s.}$ Namely, $X_{\nu_m \wedge \cdot}$ is a bounded martingale with respect to (\mathbf{F}^P, P) . Clearly, $\{\nu_m\}_{m \in \mathbb{N}}$ are \mathbf{F}^P -stopping times with $\lim_{m \rightarrow \infty} \uparrow \nu_m = T$. Hence, X is a local martingale with respect to (\mathbf{F}^P, P) . More general, any semi-martingale with respect to (\mathbf{F}^t, P) is also a semi-martingale with respect to (\mathbf{F}^P, P) .

(2) The uniqueness is obvious and it suffices to show the existence for case $\mathbb{E} = \mathbb{R}$: Let $\{X_s\}_{s \in [t, T]}$ be a real-valued, \mathbf{F}^P -adapted continuous process. For each $s \in \mathbb{Q}_{t, T}$, we see from (6.3) that

$$\tilde{X}_s \triangleq E_P[X_s | \mathcal{F}_s^t] = E_P[X_s | \mathcal{F}_s^P] = X_s, \quad P\text{-a.s.}$$

Set $\mathcal{N} \triangleq \{\omega \in \Omega^t : \text{the path } s \rightarrow X_s(\omega) \text{ is not continuous}\} \cup \left(\bigcup_{s \in \mathbb{Q}_{t, T}} \{X_s \neq \tilde{X}_s\}\right) \in \mathcal{N}^P$. Since $X_s^n \triangleq \sum_{i=1}^{1+[n(T-t)]} \tilde{X}_{t+\frac{i-1}{n}} \mathbf{1}_{\{s \in [t+\frac{i-1}{n}, t+\frac{i}{n}]\}}, s \in [t, T]$ is a real-valued, \mathbf{F}^t -progressively measurable process for any $n \in \mathbb{N}$, We see that $\tilde{X}_s \triangleq \left(\overline{\lim}_{n \rightarrow \infty} X_s^n\right) \mathbf{1}_{\{\overline{\lim}_{n \rightarrow \infty} X_s^n < \infty\}}$ also defines a real-valued, \mathbf{F}^t -progressively measurable process.

Let $\omega \in \mathcal{N}^c$ and $s \in [t, T]$. For any $n \in \mathbb{N}$, since $s \in [s_n, s_n + \frac{1}{n})$ with $s_n \triangleq t + \frac{\lfloor n(s-t) \rfloor}{n}$, one has $X_s^n(\omega) = \tilde{X}_{s_n}(\omega) = X_{s_n}(\omega)$. Clearly, $\lim_{n \rightarrow \infty} \uparrow s_n = s$. As $n \rightarrow \infty$, the continuity of X shows that $\lim_{n \rightarrow \infty} X_s^n(\omega) = \lim_{n \rightarrow \infty} X_{s_n}(\omega) = X_s(\omega)$, which implies that $\mathcal{N}^c \subset \{\omega \in \Omega^t : X_s(\omega) = \tilde{X}_s(\omega), \forall s \in [t, T]\}$. Therefore, \tilde{X} is P -indistinguishable from X , and it follows that \tilde{X} also has P -a.s. continuous paths.

Next, let $\{X_s\}_{s \in [t, T]}$ be a real-valued, \mathbf{F}^P -progressively measurable process that is bounded. Since $\mathcal{X}_s \triangleq \int_t^s X_r dr, s \in [t, T]$ defines a real-valued, \mathbf{F}^P -adapted continuous process, we know from part (1) that \mathcal{X} has a unique \mathbf{F}^t -version $\tilde{\mathcal{X}}$. For any $n \in \mathbb{N}$, $X_s^n \triangleq n(\tilde{\mathcal{X}}_s - \tilde{\mathcal{X}}_{(s-1/n) \vee t})$ is clearly a real-valued, \mathbf{F}^t -adapted continuous process and thus an \mathbf{F}^t -progressively measurable process. It follows that $\tilde{X}_s \triangleq \left(\overline{\lim}_{n \rightarrow \infty} X_s^n\right) \mathbf{1}_{\{\overline{\lim}_{n \rightarrow \infty} X_s^n < \infty\}}$ again defines a real-valued, \mathbf{F}^t -progressively measurable process.

Set $\hat{\mathcal{N}} \triangleq \{\omega \in \Omega^t : \mathcal{X}_s(\omega) \neq \tilde{\mathcal{X}}_s(\omega) \text{ for some } s \in [t, T]\} \in \mathcal{N}^P$. For any $\omega \in \hat{\mathcal{N}}^c$, one can deduce that

$$\lim_{n \rightarrow \infty} X_s^n(\omega) = \lim_{n \rightarrow \infty} n(\mathcal{X}_s - \mathcal{X}_{(s-1/n) \vee t}) = \lim_{n \rightarrow \infty} n \int_{(s-1/n) \vee t}^s X_r dr = X_s, \quad \text{for a.e. } s \in [t, T],$$

which implies that $\tilde{X}_s(\omega) = X_s(\omega)$ for $ds \times dP$ -a.s. $(s, \omega) \in [t, T] \times \Omega^t$.

Moreover, for general real-valued, \mathbf{F}^P -progressively measurable process $\{X_s\}_{s \in [t, T]}$, let \tilde{X}^m be the \mathbf{F}^t -version of $\{X_s^m = (-m) \vee (X_s \wedge m)\}_{s \in [t, T]}$ for any $m \in \mathbb{N}$. Then $\tilde{X}_s \triangleq \left(\overline{\lim}_{m \rightarrow \infty} \tilde{X}_s^m\right) \mathbf{1}_{\{\overline{\lim}_{m \rightarrow \infty} \tilde{X}_s^m < \infty\}}$ defines a real-valued, \mathbf{F}^t -progressively measurable process. Let $\mathcal{D} \triangleq \bigcup_{m \in \mathbb{N}} \{(s, \omega) \in [t, T] \times \Omega^t : X_s^m(\omega) \neq \tilde{X}_s^m(\omega)\}$. Clearly, $ds \times dP(\mathcal{D}) = 0$ and it holds for any $(s, \omega) \in ([t, T] \times \Omega^t) \setminus \mathcal{D}$ that $\tilde{X}_s(\omega) = X_s(\omega)$. \square

Lemma 6.1. *Given $t \in [0, T]$ and two (t, q) -parameter sets $(\xi_1, f_1, \underline{L}^1, \bar{L}^1)$, $(\xi_2, f_2, \underline{L}^2, \bar{L}^2)$ with $P_0^t(\underline{L}_s^1 \leq \underline{L}_s^2, \bar{L}_s^1 \leq \bar{L}_s^2, \forall s \in [t, T]) = 1$, let $(Y^i, Z^i, \underline{K}^i, \bar{K}^i) \in \mathbb{C}_{\mathbf{F}^t}^q([t, T]) \times \mathbb{H}_{\mathbf{F}^t}^{2,q}([t, T], \mathbb{R}^d) \times \mathbb{K}_{\mathbf{F}^t}([t, T]) \times \mathbb{K}_{\mathbf{F}^t}([t, T])$, $i = 1, 2$ be a solution of DRBSDE($P_0^t, \xi_i, f_i, \underline{L}^i, \bar{L}^i$). For either $i = 1$ or $i = 2$, if f_i satisfies (1.6), then for any $\varpi \in (1, q]$*

$$E_t \left[\sup_{s \in [t, T]} |(Y_s^1 - Y_s^2)^+|^{\varpi} \right] \leq C(T, \varpi, \gamma) \left\{ E_t[|(\xi_1 - \xi_2)^+|^{\varpi}] + E_t \left[\left(\int_t^T (f_1(r, Y_r^{3-i}, Z_r^{3-i}) - f_2(r, Y_r^{3-i}, Z_r^{3-i}))^+ dr \right)^{\varpi} \right] \right\}.$$

Proof: Without loss of generality, let f_1 satisfy (1.6). Fix $\varpi \in (1, q]$. We assume that

$$E_t \left[\left(\int_t^T (f_1(r, Y_r^2, Z_r^2) - f_2(r, Y_r^2, Z_r^2))^+ dr \right)^{\varpi} \right] < \infty, \quad (6.5)$$

otherwise, the result holds automatically.

For $\mathfrak{X} = \xi, Y, Z$, we set $\Delta \mathfrak{X} \triangleq \mathfrak{X}^1 - \mathfrak{X}^2$. Applying Tanaka's formula to process $(\Delta Y)^+$ yields that

$$\begin{aligned} (\Delta Y_s)^+ &= (\Delta \xi)^+ + \int_s^T \mathbf{1}_{\{\Delta Y_r > 0\}} (f_1(r, Y_r^1, Z_r^1) - f_2(r, Y_r^2, Z_r^2)) dr - \frac{1}{2} \int_s^T d\mathfrak{L}_r \\ &\quad + \int_s^T \mathbf{1}_{\{\Delta Y_r > 0\}} (d\underline{K}_r^1 - d\underline{K}_r^2 - d\bar{K}_r^1 + d\bar{K}_r^2) - \int_s^T \mathbf{1}_{\{\Delta Y_r > 0\}} \Delta Z_r dB_r^t, \quad \forall t \leq s \leq T, \end{aligned}$$

where \mathfrak{L} is a real-valued, \mathbf{F}^t -adapted, increasing and continuous process known as “local time”. Then we can deduce from Lemma 2.1 of [15] that

$$\begin{aligned} &|(\Delta Y_s)^+|^{\varpi} - |(\Delta Y_{s'})^+|^{\varpi} + \frac{\varpi(\varpi - 1)}{2} \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_s)^+|^{\varpi-2} |\Delta Z_r|^2 dr \\ &\leq \varpi \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} (f_1(r, Y_r^1, Z_r^1) - f_2(r, Y_r^2, Z_r^2)) dr + \varpi \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} (d\underline{K}_r^1 - d\underline{K}_r^2 - d\bar{K}_r^1 + d\bar{K}_r^2) \\ &\quad - \varpi \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} \Delta Z_r dB_r^t - \frac{\varpi}{2} \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} d\mathfrak{L}_r, \quad \forall t \leq s \leq s' \leq T. \end{aligned} \quad (6.6)$$

By the lower flat-off condition of DRBSDE($P_0^t, \xi_1, f_1, \underline{L}^1, \bar{L}^1$), it holds P_0^t -a.s. that

$$0 \leq \int_t^T \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} d\underline{K}_r^1 = \int_t^T \mathbf{1}_{\{\underline{L}_r^1 = Y_r^1 > Y_r^2\}} |(\underline{L}_r^1 - Y_r^2)^+|^{\varpi-1} d\underline{K}_r^1 \leq \int_t^T \mathbf{1}_{\{\underline{L}_r^1 > \underline{L}_r^2\}} |(\underline{L}_r^1 - Y_r^2)^+|^{\varpi-1} d\underline{K}_r^1 = 0.$$

Similarly, the upper flat-off condition of DRBSDE($P_0^t, \xi_2, f_2, \underline{L}^2, \bar{L}^2$) implies that P_0^t -a.s.

$$0 \leq \int_t^T \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} d\bar{K}_r^2 = \int_t^T \mathbf{1}_{\{Y_r^1 > Y_r^2 = \bar{L}_r^2\}} (Y_r^1 - \bar{L}_r^2)^+ |(\bar{L}_r^2 - Y_r^2)^+|^{\varpi-1} d\bar{K}_r^2 \leq \int_t^T \mathbf{1}_{\{\bar{L}_r^1 > \bar{L}_r^2\}} (Y_r^1 - \bar{L}_r^2)^+ |(\bar{L}_r^2 - Y_r^2)^+|^{\varpi-1} d\bar{K}_r^2 = 0.$$

Putting these two inequality back into (6.6) and using Lipschitz continuity of f_1 in (y, z) , we obtain

$$\begin{aligned} &|(\Delta Y_s)^+|^{\varpi} + \frac{\varpi(\varpi - 1)}{2} \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_s)^+|^{\varpi-2} |\Delta Z_r|^2 dr \\ &\leq |(\Delta Y_{s'})^+|^{\varpi} + \varpi \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} \left[(\gamma |\Delta Y_r| + \gamma |\Delta Z_r| + \Delta f_+) dr - \Delta Z_r dB_r^t \right], \quad \forall t \leq s \leq s' \leq T, \end{aligned} \quad (6.7)$$

where $\Delta f_+ \triangleq (f_1(r, Y_r^2, Z_r^2) - f_2(r, Y_r^2, Z_r^2))^+$.

Since $\mathbb{C}_{\mathbf{F}^t}^q([t, T]) \subset \mathbb{C}_{\mathbf{F}^t}^{\varpi}([t, T])$ and $\mathbb{H}_{\mathbf{F}^t}^{2,q}([t, T], \mathbb{R}^d) \subset \mathbb{H}_{\mathbf{F}^t}^{2,\varpi}([t, T], \mathbb{R}^d)$ by Jensen's inequality, the Burkholder-Davis-Gundy inequality and Hölder's inequality imply that for some $c > 0$

$$\begin{aligned} E_t \left[\sup_{s \in [t, T]} \left| \int_t^s \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} \Delta Z_r dB_r^t \right| \right] &\leq c E_t \left[\left(\int_t^T |(\Delta Y_r)^+|^{2\varpi-2} |\Delta Z_r|^2 dr \right)^{1/2} \right] \\ &\leq c E_t \left[\sup_{r \in [t, T]} |\Delta Y_r|^{\varpi-1} \left(\int_t^T |\Delta Z_r|^2 dr \right)^{1/2} \right] \leq c \|\Delta Y\|_{\mathbb{C}_{\mathbf{F}^t}^{\varpi}([t, T])}^{\varpi-1} \|\Delta Z\|_{\mathbb{H}_{\mathbf{F}^t}^{2,\varpi}([t, T], \mathbb{R}^d)} < \infty, \end{aligned}$$

which shows that

$$\left\{ \int_t^s \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} \Delta Z_r dB_r^t \right\}_{s \in [t, T]} \text{ is a uniformly integrable martingale with respect to } (\bar{\mathbf{F}}^t, P_0^t). \quad (6.8)$$

Then, letting $s = t$, $s' = T$ and taking the expectation E_t in (6.7), we can deduce from Hölder's inequality, Young's inequality and (6.5) that

$$\begin{aligned} & \frac{\varpi(\varpi-1)}{2} E_t \int_t^T \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-2} |\Delta Z_r|^2 dr \\ & \leq E_t [|\Delta \xi|^\varpi] + \varpi E_t \left[\left(\sup_{r \in [t, T]} |\Delta Y_r|^{\varpi-1} \right) \left(\gamma(T-t) \sup_{r \in [t, T]} |\Delta Y_r| + \gamma \sqrt{T-t} \left(\int_t^T |\Delta Z_r|^2 dr \right)^{1/2} + \int_t^T \Delta \mathbf{f}_+ dr \right) \right] \\ & \leq \varpi(1 + \gamma(T-t)) \|\Delta Y\|_{\mathbb{C}_{\bar{\mathbf{F}}^t}^\varpi([t, T])}^\varpi + \gamma^\varpi(T-t)^{\varpi/2} \|\Delta Z\|_{\mathbb{H}_{\bar{\mathbf{F}}^t}^{2, \varpi}([t, T], \mathbb{R}^d)}^\varpi + E_t \left[\left(\int_t^T \Delta \mathbf{f}_+ dr \right)^\varpi \right] < \infty. \end{aligned}$$

Hence, we can define an increasing sequence of \mathbf{F}^t -stopping times

$$\tau_n \triangleq \inf \left\{ s \in [t, T] : \int_t^s \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-2} |\Delta Z_r|^2 dr > n \right\} \wedge T, \quad \forall n \in \mathbb{N}$$

such that $\lim_{n \rightarrow \infty} \uparrow \tau_n = T$, P_0^t -a.s. Fix $n \in \mathbb{N}$. Since

$$\mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} |\Delta Z_r| \leq \frac{\gamma}{\varpi-1} |(\Delta Y_r)^+|^\varpi + \frac{\varpi-1}{4\gamma} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-2} |\Delta Z_r|^2, \quad s \in [t, T],$$

letting $s = \tau_n \wedge s$ and $s' = \tau_n$ in (6.7) yields that

$$\begin{aligned} & |(\Delta Y_{\tau_n \wedge s})^+|^\varpi + \frac{\varpi(\varpi-1)}{4} \int_{\tau_n \wedge s}^{\tau_n} |(\Delta Y_r)^+|^{\varpi-2} \mathbf{1}_{\{\Delta Y_r > 0\}} |\Delta Z_r|^2 dr \leq |(\Delta Y_{\tau_n})^+|^\varpi + \varpi \int_{\tau_n \wedge s}^{\tau_n} |(\Delta Y_r)^+|^{\varpi-1} \Delta \mathbf{f}_+ dr \\ & + \left(\varpi\gamma + \frac{\varpi\gamma^2}{\varpi-1} \right) \int_{\tau_n \wedge s}^{\tau_n} |(\Delta Y_r)^+|^\varpi dr - \varpi \int_{\tau_n \wedge s}^{\tau_n} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} \Delta Z_r dB_r^t, \quad s \in [t, T]. \end{aligned} \quad (6.9)$$

Taking the expectation E_t , we can deduce from Fubini's Theorem, (6.8) and Optional Sampling Theorem that

$$\begin{aligned} & E_t \left[|(\Delta Y_{\tau_n \wedge s})^+|^\varpi \right] + \frac{\varpi(\varpi-1)}{4} E_t \int_{\tau_n \wedge s}^{\tau_n} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-2} |\Delta Z_r|^2 dr \\ & \leq E_t [\eta_n] + \left(\varpi\gamma + \frac{\varpi\gamma^2}{\varpi-1} \right) \int_s^T E_t \left[|(\Delta Y_{\tau_n \wedge r})^+|^\varpi \right] dr, \quad s \in [t, T], \end{aligned} \quad (6.10)$$

where $\eta_n \triangleq |(\Delta Y_{\tau_n})^+|^\varpi + \varpi \int_t^{\tau_n} |(\Delta Y_r)^+|^{\varpi-1} \Delta \mathbf{f}_+ dr$.

Let $C(T, \varpi, \gamma)$ denote a generic constant, depending on T, ϖ, γ , whose form may vary from line to line. An application of Gronwall's inequality to (6.10) yields that

$$E_t \left[|(\Delta Y_{\tau_n \wedge s})^+|^\varpi \right] + \frac{\varpi(\varpi-1)}{4} E_t \int_{\tau_n \wedge s}^{\tau_n} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-2} |\Delta Z_r|^2 dr \leq C(T, \varpi, \gamma) E_t [\eta_n], \quad s \in [t, T]. \quad (6.11)$$

which together with Fubini's Theorem shows that

$$E_t \int_t^{\tau_n} |(\Delta Y_s)^+|^\varpi ds \leq E_t \int_t^T |(\Delta Y_{\tau_n \wedge s})^+|^\varpi ds = \int_t^T E_t \left[|(\Delta Y_{\tau_n \wedge s})^+|^\varpi \right] ds \leq C(T, \varpi, \gamma) E_t [\eta_n].$$

Then we can deduce from (6.9) that

$$E_t \left[\sup_{s \in [t, \tau_n]} |(\Delta Y_s)^+|^\varpi \right] \leq C(T, \varpi, \gamma) E_t [\eta_n] + \varpi E_t \left[\sup_{s \in [t, T]} \left| \int_{\tau_n \wedge s}^{\tau_n} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} \Delta Z_r dB_r^t \right| \right].$$

The Burkholder-Davis-Gundy inequality again implies that for some $c > 0$

$$\begin{aligned} E_t \left[\sup_{s \in [t, T]} \left| \int_s^T \mathbf{1}_{\{r \leq \tau_n\}} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-1} \Delta Z_r dB_r^t \right| \right] &\leq c E_t \left[\left(\int_t^T \mathbf{1}_{\{r \leq \tau_n\}} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{2\varpi-2} |\Delta Z_r|^2 dr \right)^{1/2} \right] \\ &\leq c E_t \left[\sup_{r \in [t, \tau_n]} |(\Delta Y_r)^+|^{\varpi/2} \left(\int_t^{\tau_n} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-2} |\Delta Z_r|^2 dr \right)^{1/2} \right] \leq \frac{1}{2\varpi} E_t \left[\sup_{r \in [t, \tau_n]} |(\Delta Y_r)^+|^{\varpi} \right] \\ &\quad + \frac{\varpi c^2}{2} E_t \int_t^{\tau_n} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{\varpi-2} |\Delta Z_r|^2 dr \leq \frac{1}{2\varpi} E_t \left[\sup_{r \in [t, \tau_n]} |(\Delta Y_r)^+|^{\varpi} \right] + C(T, \varpi, \gamma) E_t [\eta_n], \end{aligned}$$

where we used (6.11) with $s = t$ in the last inequality. Since $E_t \left[\sup_{r \in [t, \tau_n]} |(\Delta Y_r)^+|^{\varpi} \right] \leq \|\Delta Y\|_{\mathbb{C}_{\mathbf{F}^t}^{\varpi}([t, T])}^{\varpi} < \infty$, it follows from Young's inequality that

$$\begin{aligned} E_t \left[\sup_{s \in [t, \tau_n]} |(\Delta Y_s)^+|^{\varpi} \right] &\leq C(T, \varpi, \gamma) E_t [\eta_n] \leq C(T, \varpi, \gamma) \left\{ E_t \left[|(\Delta Y_{\tau_n})^+|^{\varpi} \right] + E_t \left[\sup_{r \in [t, \tau_n]} |(\Delta Y_r)^+|^{\varpi-1} \int_t^{\tau_n} \Delta f_+ dr \right] \right\} \\ &\leq C(T, \varpi, \gamma) E_t \left[|(\Delta Y_{\tau_n})^+|^{\varpi} \right] + \frac{1}{2} E_t \left[\sup_{r \in [t, \tau_n]} |(\Delta Y_r)^+|^{\varpi} \right] + C(T, \varpi, \gamma) E_t \left[\left(\int_t^{\tau_n} \Delta f_+ dr \right)^{\varpi} \right]. \end{aligned}$$

Hence, we have

$$E_t \left[\sup_{s \in [t, \tau_n]} |(\Delta Y_s)^+|^{\varpi} \right] \leq C(T, \varpi, \gamma) \left\{ E_t \left[|(\Delta Y_{\tau_n})^+|^{\varpi} \right] + E_t \left[\left(\int_t^T \Delta f_+ dr \right)^{\varpi} \right] \right\}. \quad (6.12)$$

Since $\Delta Y \in \mathbb{C}_{\mathbf{F}^t}^{\varpi}([t, T])$ and since $\lim_{n \rightarrow \infty} \uparrow \tau_n = T$, P_0^t -a.s., the Dominated Convergence Theorem implies that $\lim_{n \rightarrow \infty} E_t \left[|(\Delta Y_{\tau_n})^+|^{\varpi} \right] = E_t \left[|(\Delta \xi)^+|^{\varpi} \right]$. Letting $n \rightarrow \infty$ in (6.12) and applying the Monotone Convergence Theorem on its left-hand-side lead to that

$$E_t \left[\sup_{s \in [t, T]} |(Y_s^1 - Y_s^2)^+|^{\varpi} \right] \leq C(T, \varpi, \gamma) \left\{ E_t \left[|(\xi_1 - \xi_2)^+|^{\varpi} \right] + E_t \left[\left(\int_t^T (f_1(r, Y_r^2, Z_r^2) - f_2(r, Y_r^2, Z_r^2))^+ dr \right)^{\varpi} \right] \right\}. \quad (6.13)$$

Proof of Proposition 1.1: For either $i = 1$ or $i = 2$, if f_i satisfies (1.6), applying Lemma 6.1 with $\varpi = q$ yields that $E_t \left[\sup_{s \in [t, T]} |(Y_s^1 - Y_s^2)^+|^q \right] = 0$. Hence, it holds P_0^t -a.s. that $(\Delta Y_s)^+ = 0$, or $Y_s^1 \leq Y_s^2$ for any $s \in [t, T]$. \square

Proof of Proposition 1.2: For any $\varpi \in (1, q]$, it follows from Lemma 6.1 that

$$E_t \left[\sup_{s \in [t, T]} |(Y_s^1 - Y_s^2)^+|^{\varpi} \right] \leq C(T, \varpi, \gamma) \left\{ E_t \left[|(\xi_1 - \xi_2)^+|^{\varpi} \right] + E_t \left[\left(\int_t^T (f_1(r, Y_r^2, Z_r^2) - f_2(r, Y_r^2, Z_r^2))^+ dr \right)^{\varpi} \right] \right\}. \quad (6.14)$$

Exchanging the order of Y^1, Y^2 and applying Lemma 6.1 again give that

$$E_t \left[\sup_{s \in [t, T]} |(Y_s^2 - Y_s^1)^+|^{\varpi} \right] \leq C(T, \varpi, \gamma) \left\{ E_t \left[|(\xi_2 - \xi_1)^+|^{\varpi} \right] + E_t \left[\left(\int_t^T (f_2(r, Y_r^2, Z_r^2) - f_1(r, Y_r^2, Z_r^2))^+ dr \right)^{\varpi} \right] \right\},$$

which together with (6.14) implies (1.7). \square

Proof of Lemma 2.1: (1) Set $\Theta = (t, x, \mu, \nu)$ and fix $s \in [t, T]$. For any $s' \in [t, s]$, one can deduce from (1.1), (2.2) and (2.1) that

$$\sup_{r \in [t, s']} |X_r^{\Theta} - x| \leq \gamma \int_t^{s'} \left(1 + |x| + \sup_{r \in [t, r]} |X_r^{\Theta} - x| + [\mu_r]_{\mathbb{U}} + [\nu_r]_{\mathbb{V}} \right) dr + \sup_{r \in [t, s']} \left| \int_t^r \sigma(r, X_r^{\Theta}, \mu_r, \nu_r) dB_r^t \right|, \quad P_0^t - a.s.$$

Then Hölders inequality, Doob's martingale inequality, (2.2), (2.1) and Fubini's Theorem imply that

$$\begin{aligned} E_t \left[\sup_{r \in [t, s']} |X_r^\Theta - x|^2 \right] &\leq c_0 E_t \int_t^{s'} \left(1 + |x|^2 + \sup_{r \in [t, r]} |X_r^\Theta - x|^2 + [\mu_r]_{\mathbb{U}}^2 + [\nu_r]_{\mathbb{V}}^2 \right) dr + c_0 E_t \int_t^{s'} |\sigma(r, X_r^\Theta, \mu_r, \nu_r)|^2 dr \\ &\leq c_0(1 + |x|^2)(s - t) + c_0 \int_t^{s'} E_t \left[\sup_{r \in [t, r]} |X_r^\Theta - x|^2 \right] dr + c_0 E_t \int_t^{s'} ([\mu_r]_{\mathbb{U}}^2 + [\nu_r]_{\mathbb{V}}^2) dr, \quad s' \in [t, s]. \end{aligned}$$

An application of Gronwall's inequality yields that

$$E_t \left[\sup_{r \in [t, s']} |X_r^\Theta - x|^2 \right] \leq c_0 e^{c_0 T} (1 + |x|^2)(s - t) + c_0 e^{c_0 T} E_t \int_t^{s'} ([\mu_r]_{\mathbb{U}}^2 + [\nu_r]_{\mathbb{V}}^2) dr, \quad s' \in [t, s].$$

Taking $s' = s$ gives (2.8).

(2) Given $x' \in \mathbb{R}^k$, we set $\Delta X_r \triangleq X_r^{t, x, \mu, \nu} - X_r^{t, x', \mu, \nu}$, $\forall r \in [t, T]$. By (2.2),

$$\sup_{r \in [t, s]} |\Delta X_r| \leq |x - x'| + \gamma \int_t^s |\Delta X_r| dr + \sup_{r \in [t, s]} \left| \int_t^r \left(\sigma(r, X_r^{t, x, \mu, \nu}, \mu_r, \nu_r) - \sigma(r, X_r^{t, x', \mu, \nu}, \mu_r, \nu_r) \right) dB_r^t \right|, \quad \forall s \in [t, T].$$

Then the Burkholder-Davis-Gundy inequality and (2.2) imply that

$$\begin{aligned} E_t \left[\sup_{r \in [t, s]} |\Delta X_r|^\varpi \right] &\leq c_\varpi |x - x'|^\varpi + c_\varpi E_t \left[\left(\int_t^s |\Delta X_r| dr \right)^\varpi + \left(\int_t^s |\Delta X_r|^2 dr \right)^{\varpi/2} \right] \\ &\leq c_\varpi |x - x'|^\varpi + c_\varpi E_t \left[\left(\int_t^s |\Delta X_r| dr \right)^\varpi + \sup_{r \in [t, s]} |\Delta X_r|^{\varpi/2} \left(\int_t^s |\Delta X_r| dr \right)^{\varpi/2} \right] \\ &\leq c_\varpi |x - x'|^\varpi + c_\varpi E_t \left[\left(\int_t^s |\Delta X_r| dr \right)^\varpi \right] + \frac{1}{2} E_t \left[\sup_{r \in [t, s]} |\Delta X_r|^\varpi \right], \quad \forall s \in [t, T]. \end{aligned}$$

As $E_t \left[\sup_{r \in [t, T]} |\Delta X_r|^\varpi \right] \leq 1 + 2E_t \left[\sup_{r \in [t, T]} |X_r^{t, x, \mu, \nu}|^2 + \sup_{r \in [t, T]} |X_r^{t, x', \mu, \nu}|^2 \right] < \infty$ by (2.7), it follows from Hölder's inequality and Fubini's Theorem that

$$E_t \left[\sup_{r \in [t, s]} |\Delta X_r|^\varpi \right] \leq c_\varpi |x - x'|^\varpi + c_\varpi E_t \int_t^s |\Delta X_r|^\varpi dr \leq c_\varpi |x - x'|^\varpi + c_\varpi \int_t^s E_t \left[\sup_{r' \in [t, r]} |\Delta X_{r'}|^\varpi \right] dr, \quad \forall s \in [t, T]. \quad (6.15)$$

An application of Gronwall's inequality yields (2.9).

(3) Next, Let us assume (2.10) for some $\lambda \in (0, 1]$. Given $\mu' \in \mathcal{U}^t$, we set $\Delta \mathcal{X}_r \triangleq X_r^{t, x, \mu, \nu} - X_r^{t, x, \mu', \nu}$, $\forall r \in [t, T]$. By (2.2) and (2.10),

$$\sup_{r \in [t, s]} |\Delta \mathcal{X}_r| \leq \gamma \int_t^s \left(|\Delta \mathcal{X}_r| + \rho_{\mathbb{U}}^\lambda(\mu_r, \mu'_r) \right) dr + \sup_{r \in [t, s]} \left| \int_t^r \left(\sigma(r, X_r^{t, x, \mu, \nu}, \mu_r, \nu_r) - \sigma(r, X_r^{t, x, \mu', \nu}, \mu'_r, \nu_r) \right) dB_r^t \right|, \quad \forall s \in [t, T].$$

Then one can deduce from the Burkholder-Davis-Gundy inequality, (2.2), (2.10) and Hölder's inequality that

$$\begin{aligned} E_t \left[\sup_{r \in [t, s]} |\Delta \mathcal{X}_r|^\varpi \right] &\leq c_\varpi E_t \left[\left(\int_t^s |\Delta \mathcal{X}_r| dr \right)^\varpi + \left(\int_t^s \rho_{\mathbb{U}}^\lambda(\mu_r, \mu'_r) dr \right)^\varpi + \left(\int_t^s \left(|\Delta \mathcal{X}_r| + \rho_{\mathbb{U}}^\lambda(\mu_r, \mu'_r) \right)^2 dr \right)^{\varpi/2} \right] \\ &\leq c_\varpi E_t \left[\left(\int_t^s |\Delta \mathcal{X}_r| dr \right)^\varpi + \left(\int_t^s \rho_{\mathbb{U}}^{2\lambda}(\mu_r, \mu'_r) dr \right)^{\varpi/2} + \sup_{r \in [t, s]} |\Delta \mathcal{X}_r|^{\varpi/2} \left(\int_t^s |\Delta \mathcal{X}_r| dr \right)^{\varpi/2} \right] \\ &\leq c_\varpi E_t \left[\left(\int_t^s |\Delta \mathcal{X}_r| dr \right)^\varpi + \left(\int_t^s \rho_{\mathbb{U}}^{2\lambda}(\mu_r, \mu'_r) dr \right)^{\varpi/2} \right] + \frac{1}{2} E_t \left[\sup_{r \in [t, s]} |\Delta \mathcal{X}_r|^\varpi \right], \quad \forall s \in [t, T]. \end{aligned}$$

Similar to (6.15), it follows from Hölder's inequality and Fubini's Theorem that

$$E_t \left[\sup_{r \in [t, s]} |\Delta \mathcal{X}_r|^\varpi \right] \leq c_\varpi \int_t^s E_t \left[\sup_{r' \in [t, r]} |\Delta \mathcal{X}_{r'}|^\varpi \right] dr + c_\varpi E_t \left[\left(\int_t^s \rho_{\mathbb{U}}^{2\lambda}(\mu_r, \mu'_r) dr \right)^{\varpi/2} \right], \quad \forall s \in [t, T].$$

Then an application of Gronwall's inequality yields (2.11). Similarly, with (2.12) we can deduce (2.13) for each $\nu' \in \mathcal{V}^t$. \square

Lemma 6.2. *Let \mathbb{M} be a separable metric space with metric $\rho_{\mathbb{M}}$. For any two \mathbb{M} -valued, \mathbf{F}^t -adapted (resp. \mathbf{F}^t -progressively measurable) processes Y, Z , the nonnegative-valued process $\rho_{\mathbb{M}}(Y, Z)$ is also \mathbf{F}^t -adapted (resp. \mathbf{F}^t -progressively measurable).*

Proof: Let $\{x_n\}_{n \in \mathbb{N}}$ be the countable dense subset of \mathbb{M} and denote by $\mathcal{B}(\mathbb{M})$ the Borel- σ -field of \mathbb{M} . We first claim that for any $y, z \in \mathbb{M}$ and $\lambda > 0$,

$$\rho_{\mathbb{M}}(y, z) < \lambda \text{ if and only if there exist } n \in \mathbb{N} \text{ and } r \in \mathbb{Q} \cap (0, \lambda) \text{ such that } \rho_{\mathbb{M}}(y, x_n) < r \text{ and } \rho_{\mathbb{M}}(x_n, z) < \lambda - r. \quad (6.16)$$

“ \Leftarrow ”: This direction is obvious due to the triangle inequality. “ \Rightarrow ”: If $\rho_{\mathbb{M}}(y, z) < \lambda$, we let r be a positive rational number that is less than $\frac{1}{2}(\lambda - \rho_{\mathbb{M}}(y, z))$. There exists an $n \in \mathbb{N}$, such that $\rho_{\mathbb{M}}(y, x_n) < r$. By the triangle inequality,

$$\rho_{\mathbb{M}}(x_n, z) \leq \rho_{\mathbb{M}}(x_n, y) + \rho_{\mathbb{M}}(y, z) < r + \rho_{\mathbb{M}}(y, z) < \lambda - r.$$

So we proved the claim (6.16).

Now, given two \mathbb{M} -valued, \mathbf{F}^t -adapted processes Y and Z , for any $s \in [t, T]$ and $\lambda > 0$, (6.16) implies that

$$\{\omega \in \Omega^t : \rho_{\mathbb{M}}(Y_s(\omega), Z_s(\omega)) < \lambda\} = \bigcup_{n \in \mathbb{N}} \bigcup_{r \in \mathbb{Q} \cap (0, \lambda)} \left(\{\omega \in \Omega^t : Y_s(\omega) \in O_r(x_n)\} \cap \{\omega \in \Omega^t : Z_s(\omega) \in O_{\lambda-r}(x_n)\} \right) \in \mathcal{F}_s^t,$$

which shows $\rho_{\mathbb{M}}(Y, Z)$ is also \mathbf{F}^t -adapted.

If Y and Z are further \mathbf{F}^t -progressively measurable, then for any $s \in [t, T]$ and $\lambda > 0$, we see from (6.16) that

$$\begin{aligned} & \{(r, \omega) \in [t, s] \times \Omega^t : \rho_{\mathbb{M}}(Y_r(\omega), Z_r(\omega)) < \lambda\} \\ &= \bigcup_{n \in \mathbb{N}} \bigcup_{r \in \mathbb{Q} \cap (0, \lambda)} \left(\{(r, \omega) \in [t, s] \times \Omega^t : Y_r(\omega) \in O_r(x_n)\} \cap \{(r, \omega) \in [t, s] \times \Omega^t : Z_r(\omega) \in O_{\lambda-r}(x_n)\} \right) \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s^t. \end{aligned}$$

Namely, $\rho_{\mathbb{M}}(Y, Z)$ is \mathbf{F}^t -progressively measurable as well. \square

Proof of Lemma 2.2: We set $\Theta \triangleq (t, x, \mu, \nu)$.

(1) For any $x' \in \mathbb{R}^k$, let $\Theta' \triangleq (t, x', \mu, \nu)$ and $\Delta X \triangleq \tilde{X}^{\Theta'} - \tilde{X}^{\Theta}$. The measurability of $(f_T^{\Theta}, \Delta X)$ and (2.5) show that

$$f_{\pm}(s, \omega, y, z) \triangleq f_T^{\Theta}(s, \omega, y, z) \pm \gamma |\Delta X_s(\omega)|^{2/q}, \quad \forall (s, \omega, y, z) \in [t, T] \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d$$

define two $\mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$ -measurable functions that are Lipschitz continuous in (y, z) with coefficient γ . We see from Hölder inequality, (2.16) and (2.7) that

$$E_t \left[\left(\int_t^T |f_{\pm}(s, 0, 0)| ds \right)^q \right] \leq c_0 E_t \left[\left(\int_t^T |f_T^{\Theta}(s, 0, 0)| ds \right)^q + \sup_{s \in [t, T]} |\Delta X_s|^2 \right] < \infty. \quad (6.17)$$

Fix $\varepsilon > 0$. The function $\phi(\mathbf{x}) \triangleq (|\mathbf{x}|^2 + \varepsilon)^{1/q}$, $\mathbf{x} \in \mathbb{R}^k$ has the following derivatives: for any $i, j \in \{1, \dots, k\}$

$$\partial_i \phi(\mathbf{x}) = \frac{2}{q} \phi^{1-q}(\mathbf{x}) \mathbf{x}_i \quad \text{and} \quad \partial_{ij}^2 \phi(\mathbf{x}) = \frac{2}{q} \phi^{1-q}(\mathbf{x}) \delta_{ij} + \frac{4}{q^2} (1-q) \phi^{1-2q}(\mathbf{x}) \mathbf{x}_i \mathbf{x}_j.$$

It is easy to estimate that

$$|\mathbf{x}|^{2/q} \leq \phi(\mathbf{x}) \leq |\mathbf{x}|^{2/q} + \varepsilon^{1/q}, \quad |D\phi(\mathbf{x})| = \frac{2}{q} \phi^{1-q}(\mathbf{x}) |\mathbf{x}| \leq \frac{2}{q} |\mathbf{x}|^{\frac{2}{q}-1}, \quad \forall \mathbf{x} \in \mathbb{R}^k. \quad (6.18)$$

For any $\mathbf{z} \in \mathbb{R}^{k \times d}$, since $\text{trace}(D^2 \phi(\mathbf{x}) \mathbf{z} \mathbf{z}^T) = \frac{2}{q} \phi^{1-q}(\mathbf{x}) |\mathbf{z}|^2 + \frac{4}{q^2} (1-q) \phi^{1-2q}(\mathbf{x}) |\mathbf{z}^T \mathbf{x}|^2$, we also have

$$-\frac{2}{q} |\mathbf{x}|^{\frac{2}{q}-2} |\mathbf{z}|^2 \leq \frac{4}{q^2} (1-q) |\mathbf{x}|^{\frac{2}{q}-2} |\mathbf{z}|^2 \leq \text{trace}(D^2 \phi(\mathbf{x}) \mathbf{z} \mathbf{z}^T) \leq \frac{2}{q} |\mathbf{x}|^{\frac{2}{q}-2} |\mathbf{z}|^2, \quad \forall \mathbf{x} \in \mathbb{R}^k. \quad (6.19)$$

Let us define \mathcal{F}_T^t -measurable random variables $\xi_{\pm} \triangleq h(\tilde{X}_T^{\Theta}) \pm \gamma\phi(\Delta X_T)$ as well as real-valued, \mathbf{F}^t -adapted continuous processes

$$\underline{L}_s^{\pm} \triangleq \underline{L}_s^{\Theta} \pm \gamma\phi(\Delta X_s) \quad \text{and} \quad \overline{L}_s^{\pm} \triangleq \overline{L}_s^{\Theta} \pm \gamma\phi(\Delta X_s), \quad \forall s \in [t, T].$$

Clearly, $\underline{L}_s^{\pm} < \overline{L}_s^{\pm}$ for any $s \in [t, T]$ and $\underline{L}_T^{\pm} \leq \xi_{\pm} \leq \overline{L}_T^{\pm}$, P_0^t -a.s. Since

$$E_t \left[\sup_{s \in [t, T]} |\phi(\Delta X_s)|^q \right] \leq c_0 E_t \left[\sup_{s \in [t, T]} |\Delta X_s|^2 \right] + c_0 \varepsilon < \infty \quad (6.20)$$

by (6.18) and (2.7), we see from (2.16) that $\underline{L}^{\pm}, \overline{L}^{\pm} \in \mathbb{C}_{\mathbf{F}^t}^q([t, T])$ and it follows that $\xi_{\pm} \in \mathbb{L}^q(\mathcal{F}_T^t)$. Then Theorem 4.1 of [15] shows that the DRBSDE $(P_0^t, \xi_{\pm}, f_{\pm}, \underline{L}^{\pm}, \overline{L}^{\pm})$ admits a unique solution $(Y^{\pm}, Z^{\pm}, \underline{K}^{\pm}, \overline{K}^{\pm}) \in \mathbb{G}_{\mathbf{F}^t}^q([t, T])$. For any $(s, \omega, y, z) \in [t, T] \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d$, the $2/q$ -Hölder continuity of h , (2.5) and (2.3) imply that

$$\begin{aligned} \xi_-(\omega) &\leq h(\tilde{X}_T^{\Theta'}(\omega)) \leq \xi_+(\omega), \quad f_-(s, \omega, y, z) \leq f_T^{\Theta'}(s, \omega, y, z) \leq f_+(s, \omega, y, z), \\ \underline{L}_s^-(\omega) &\leq \underline{L}_s^{\Theta'}(\omega) \leq \underline{L}_s^+(\omega), \quad \text{and} \quad \overline{L}_s^-(\omega) \leq \overline{L}_s^{\Theta'}(\omega) \leq \overline{L}_s^+(\omega). \end{aligned}$$

Clearly, these inequalities also hold if Θ' is replaced by Θ . Proposition 1.1 then yields that P_0^t -a.s.

$$Y_s^- \leq Y_s^{\Theta'}(T, h(\tilde{X}_T^{\Theta'})) \leq Y_s^+, \quad \text{and} \quad Y_s^- \leq Y_s^{\Theta}(T, h(\tilde{X}_T^{\Theta})) \leq Y_s^+, \quad \forall s \in [t, T]. \quad (6.21)$$

By (6.20), the processes $\hat{Y}_s^{\pm} \triangleq Y_s^{\pm} \mp \gamma\phi(\Delta X_s)$, $s \in [t, T]$ are of $\mathbb{C}_{\mathbf{F}^t}^q([t, T])$. Applying Itô's formula yields that

$$\begin{aligned} \hat{Y}_s^{\pm} &= \xi_{\pm} + \int_s^T \left[f_{\pm}(r, Y_r^{\pm}, Z_r^{\pm}) \pm \gamma((D\phi)(\Delta X_r))^T \Delta b_r \pm \frac{1}{2} \gamma \text{trace}(D^2\phi(\Delta X_r) \Delta\sigma_r (\Delta\sigma_r)^T) \right] dr \\ &\quad + \underline{K}_T^{\pm} - \underline{K}_s^{\pm} - (\overline{K}_T^{\pm} - \overline{K}_s^{\pm}) - \int_s^T \hat{Z}_r^{\pm} dB_r^t, \quad s \in [t, T], \end{aligned} \quad (6.22)$$

where $\Delta b_r \triangleq b(r, \tilde{X}_r^{\Theta'}, \mu_r, \nu_r) - b(r, \tilde{X}_r^{\Theta}, \mu_r, \nu_r)$, $\Delta\sigma_r \triangleq \sigma(r, \tilde{X}_r^{\Theta'}, \mu_r, \nu_r) - \sigma(r, \tilde{X}_r^{\Theta}, \mu_r, \nu_r)$ and $\hat{Z}_r^{\pm} \triangleq (Z_r^{\pm} \mp \gamma((D\phi)(\Delta X_r))^T \Delta\sigma_r)$. To wit, $(\hat{Y}^{\pm}, \hat{Z}^{\pm}, \underline{K}^{\pm}, \overline{K}^{\pm})$ solves the DRBSDE $(P_0^t, \xi_{\pm}, \hat{f}^{\pm}, \underline{L}^{\Theta}, \overline{L}^{\Theta})$ with

$$\begin{aligned} \hat{f}^{\pm}(s, \omega, y, z) &\triangleq f_{\pm}(s, \omega, y \pm \gamma\phi(\Delta X_s(\omega)), z \pm \gamma((D\phi)(\Delta X_s(\omega)))^T \Delta\sigma_s(\omega)) \pm \gamma((D\phi)(\Delta X_r(\omega)))^T \Delta b_r(\omega) \\ &\quad \pm \frac{1}{2} \gamma \text{trace}(D^2\phi(\Delta X_r(\omega)) \Delta\sigma_r(\omega) (\Delta\sigma_r(\omega))^T), \quad \forall (s, \omega, y, z) \in [t, T] \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d. \end{aligned}$$

The measurability of $(f_{\pm}, b, \sigma, \tilde{X}^{\Theta'}, \tilde{X}^{\Theta}, \mu, \nu)$ and the Lipschitz continuity of f_{\pm} in (y, z) imply that \hat{f}^{\pm} are also $\mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) / \mathcal{B}(\mathbb{R})$ -measurable functions that are Lipschitz continuous in (y, z) with coefficient γ . Then we can deduce from (6.19), (6.18), (2.2) that

$$\begin{aligned} |\hat{f}^{\pm}(s, 0, 0)| &\leq |f_{\pm}(s, 0, 0)| + \gamma^2 \phi(\Delta X_s) + |((D\phi)(\Delta X_s))|(\gamma^2 |\Delta\sigma_s| + \gamma |\Delta b_s|) + \frac{\gamma}{q} |\Delta X_s|^{\frac{2}{q}-2} |\Delta\sigma_r|^2 \\ &\leq |f_{\pm}(s, 0, 0)| + c_0 |\Delta X_s|^{2/q} + c_0 \varepsilon^{1/q}, \quad s \in [t, T]. \end{aligned} \quad (6.23)$$

Similarly, one can deduce from (6.18) (2.2), and (2.7) that

$$E_t \left[\left(\int_t^T |((D\phi)(\Delta X_s))^T \Delta\sigma_s|^2 ds \right)^{q/2} \right] \leq c_0 E_t \left[\left(\int_t^T |\Delta X_s|^{4/q} ds \right)^{q/2} \right] \leq c_0 E_t \left[\sup_{s \in [t, T]} |\Delta X_s|^2 \right] < \infty,$$

which shows that $\hat{Z}^{\pm} \in \mathbb{H}_{\mathbf{F}^t}^{2,q}([t, T], \mathbb{R}^d)$. Then Proposition 1.2 implies that

$$E_t \left[\sup_{s \in [t, T]} |\hat{Y}_s^+ - \hat{Y}_s^-|^{\varpi} \right] \leq c_{\varpi} \left\{ E_t[|\xi_+ - \xi_-|^{\varpi}] + E_t \left[\left(\int_t^T |\hat{f}^+(r, \hat{Y}_r^-, \hat{Z}_r^-) - \hat{f}^-(r, \hat{Y}_r^-, \hat{Z}_r^-)| dr \right)^{\varpi} \right] \right\}. \quad (6.24)$$

Since

$$\begin{aligned} \hat{f}^+(s, \hat{Y}_s^-, \hat{Z}_s^-) - \hat{f}^-(s, \hat{Y}_s^-, \hat{Z}_s^-) &= f_T^\Theta(s, \omega, \hat{Y}_s^- + \gamma\phi(\Delta X_s), \hat{Z}_s^- + \gamma((D\phi)(\Delta X_s))^T \Delta\sigma_s) \\ &\quad - f_T^\Theta(s, \omega, \hat{Y}_s^- - \gamma\phi(\Delta X_s), \hat{Z}_s^- - \gamma((D\phi)(\Delta X_s))^T \Delta\sigma_s) + 2\gamma|\Delta X_s|^{2/q} \\ &\quad + 2\gamma((D\phi)(\Delta X_s))^T \Delta b_s + \gamma \text{trace}(D^2\phi(\Delta X_s)\Delta\sigma_s(\Delta\sigma_s)^T), \quad s \in [t, T], \end{aligned}$$

similar to (6.23), the Lipschitz continuity of f_T^Θ , (6.19), (6.18) and (2.2) imply that

$$|\hat{f}^+(s, \hat{Y}_s^-, \hat{Z}_s^-) - \hat{f}^-(s, \hat{Y}_s^-, \hat{Z}_s^-)| \leq c_0|\Delta X_s|^{2/q} + c_0\varepsilon^{1/q}, \quad s \in [t, T].$$

Putting this back into (6.24), we can deduce from (6.21), (6.18) and (2.9) that

$$\begin{aligned} E_t \left[\sup_{s \in [t, T]} |Y_s^{\Theta'}(T, h(\tilde{X}_T^{\Theta'})) - Y_s^\Theta(T, h(\tilde{X}_T^\Theta))|^\varpi \right] &\leq E_t \left[\sup_{s \in [t, T]} |Y_s^+ - Y_s^-|^\varpi \right] \leq c_\varpi E_t \left[\sup_{s \in [t, T]} |\hat{Y}_s^+ - \hat{Y}_s^-|^\varpi \right] \\ &\quad + c_\varpi E_t \left[\sup_{s \in [t, T]} |\phi(\Delta X_s)|^\varpi \right] \leq c_\varpi \left\{ E_t \left[\sup_{s \in [t, T]} |\Delta X_s|^{\frac{2\varpi}{q}} \right] + \varepsilon^{\frac{\varpi}{q}} \right\} \leq c_\varpi (|x' - x|^{\frac{2\varpi}{q}} + \varepsilon^{\frac{\varpi}{q}}). \end{aligned}$$

Then letting $\varepsilon \rightarrow 0$ yields (2.19).

(2) Next, we assume that \underline{l}, \bar{l} and h satisfy (2.20), that b, σ are λ -Hölder continuous in u , and that f is 2λ -Hölder continuous in u for some $\lambda \in (0, 1/q]$. For any $\mu' \in \mathcal{U}^t$, let $\Theta^* \triangleq (t, x, \mu', \nu)$ and $\Delta\mathcal{X} \triangleq \tilde{X}^{\Theta^*} - \tilde{X}^\Theta$. The measurability of $(f_T^\Theta, \Delta\mathcal{X}, \mu', \mu)$ together with Lemma 6.2 and (2.5) shows that

$$\mathfrak{f}_\pm(s, \omega, y, z) \triangleq f_T^\Theta(s, \omega, y, z) \pm \gamma|\Delta\mathcal{X}_s(\omega)|^{2/q} \pm \gamma\rho_{\mathbb{U}}^{2/q}(\mu'_s(\omega), \mu_s(\omega)), \quad \forall (s, \omega, y, z) \in [t, T] \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d$$

define two $\mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$ -measurable functions that are Lipschitz continuous in (y, z) with coefficient γ . Similar to (6.17), Hölder inequality, (2.16) and (2.7) imply that

$$E_t \left[\left(\int_t^T |\mathfrak{f}_\pm(s, 0, 0)| ds \right)^q \right] \leq c_0 E_t \left[\left(\int_t^T |f_T^\Theta(s, 0, 0)| ds \right)^q + \sup_{s \in [t, T]} |\Delta\mathcal{X}_s|^2 + \int_t^T ([\mu'_s]_{\mathbb{U}}^2 + [\mu_s]_{\mathbb{U}}^2) ds \right] < \infty.$$

The function $\bar{\psi}(\mathfrak{x}) = \psi(|x|)$, $\mathfrak{x} \in \mathbb{R}^k$ has the following derivatives: for any $i, j \in \{1, \dots, k\}$

$$\begin{aligned} \partial_i \bar{\psi}(\mathfrak{x}) &= \mathbf{1}_{\{|\mathfrak{x}| < R_1\}} \mathfrak{x}_i + \mathbf{1}_{\{|\mathfrak{x}| \in [R_1, R_2]\}} \psi'(|\mathfrak{x}|) |\mathfrak{x}|^{-1} \mathfrak{x}_i + \mathbf{1}_{\{|\mathfrak{x}| > R_2\}} \frac{2}{q} |\mathfrak{x}|^{\frac{2}{q}-2} \mathfrak{x}_i \\ \text{and } \partial_{ij}^2 \bar{\psi}(\mathfrak{x}) &= \mathbf{1}_{\{|\mathfrak{x}| < R_1\}} \delta_{ij} + \mathbf{1}_{\{|\mathfrak{x}| \in [R_1, R_2]\}} (\psi'(|\mathfrak{x}|) |\mathfrak{x}|^{-1} \delta_{ij} - |\mathfrak{x}|^{-3} \mathfrak{x}_i \mathfrak{x}_j + \psi''(|\mathfrak{x}|) |\mathfrak{x}|^{-2} \mathfrak{x}_i \mathfrak{x}_j) \\ &\quad + \mathbf{1}_{\{|\mathfrak{x}| > R_2\}} \left(\frac{2}{q} |\mathfrak{x}|^{\frac{2}{q}-2} \delta_{ij} + \frac{4}{q^2} (1-q) |\mathfrak{x}|^{\frac{2}{q}-4} \mathfrak{x}_i \mathfrak{x}_j \right). \end{aligned}$$

We can estimate that

$$|D\bar{\psi}(\mathfrak{x})| = \mathbf{1}_{\{|\mathfrak{x}| < R_1\}} |\mathfrak{x}| + \mathbf{1}_{\{|\mathfrak{x}| \in [R_1, R_2]\}} \psi'(|\mathfrak{x}|) + \mathbf{1}_{\{|\mathfrak{x}| > R_2\}} \frac{2}{q} |\mathfrak{x}|^{\frac{2}{q}-1} \leq \kappa_\psi + \frac{2}{q} |\mathfrak{x}|^{\frac{2}{q}-1}, \quad \forall \mathfrak{x} \in \mathbb{R}^k. \quad (6.25)$$

For any $\mathfrak{z} \in \mathbb{R}^{k \times d}$, since

$$\begin{aligned} \text{trace}(D^2 \bar{\psi}(\mathfrak{x}) \mathfrak{z} \mathfrak{z}^T) &= \mathbf{1}_{\{|\mathfrak{x}| < R_1\}} |\mathfrak{z}|^2 + \mathbf{1}_{\{|\mathfrak{x}| \in [R_1, R_2]\}} (\psi'(|\mathfrak{x}|) |\mathfrak{x}|^{-1} |\mathfrak{z}|^2 - |\mathfrak{x}|^{-3} |\mathfrak{z}^T \mathfrak{x}|^2 + \psi''(|\mathfrak{x}|) |\mathfrak{x}|^{-2} |\mathfrak{z}^T \mathfrak{x}|^2) \\ &\quad + \mathbf{1}_{\{|\mathfrak{x}| > R_2\}} \left(\frac{2}{q} |\mathfrak{x}|^{\frac{2}{q}-2} |\mathfrak{z}|^2 + \frac{4}{q^2} (1-q) |\mathfrak{x}|^{\frac{2}{q}-4} |\mathfrak{z}^T \mathfrak{x}|^2 \right), \end{aligned}$$

similar to (6.19), one can show that

$$\begin{aligned} |\text{trace}(D^2 \bar{\psi}(\mathfrak{x}) \mathfrak{z} \mathfrak{z}^T)| &\leq \mathbf{1}_{\{|\mathfrak{x}|=0\}} |\mathfrak{z}|^2 + \mathbf{1}_{\{0 < |\mathfrak{x}| \leq R_2\}} \left(1 + R_1^{-1} \sup_{\lambda \in [R_1, R_2]} 1 \vee \psi'(\lambda) + \sup_{\lambda \in [R_1, R_2]} |\psi''(\lambda)| \right) |\mathfrak{z}|^2 + \mathbf{1}_{\{|\mathfrak{x}| > R_2\}} \frac{2}{q} |\mathfrak{x}|^{\frac{2}{q}-2} |\mathfrak{z}|^2 \\ &\leq \mathbf{1}_{\{|\mathfrak{x}|=0\}} |\mathfrak{z}|^2 + \mathbf{1}_{\{|\mathfrak{x}| > 0\}} \kappa_\psi (1 \wedge |\mathfrak{x}|^{\frac{2}{q}-2}) |\mathfrak{z}|^2, \quad \forall \mathfrak{x} \in \mathbb{R}^k. \end{aligned} \quad (6.26)$$

Let us define \mathcal{F}_T^t -measurable random variables $\eta_{\pm} \triangleq h(\tilde{X}_T^{\Theta}) \pm \gamma \bar{\psi}(\Delta \mathcal{X}_T)$ as well as real-valued, \mathbf{F}^t -adapted continuous processes

$$\underline{\mathcal{L}}_s^{\pm} \triangleq \underline{L}_s^{\Theta} \pm \gamma \bar{\psi}(\Delta \mathcal{X}_s) \quad \text{and} \quad \overline{\mathcal{L}}_s^{\pm} \triangleq \overline{L}_s^{\Theta} \pm \gamma \bar{\psi}(\Delta \mathcal{X}_s), \quad \forall s \in [t, T].$$

Clearly, $\underline{\mathcal{L}}_s^{\pm} < \overline{\mathcal{L}}_s^{\pm}$ for any $s \in [t, T]$ and $\underline{\mathcal{L}}_T^{\pm} \leq \eta_{\pm} \leq \overline{\mathcal{L}}_T^{\pm}$, P_0^t -a.s. Since $\psi(\lambda) \leq \lambda^{2/q}$ for any $\lambda \geq 0$, we see from (2.7) that

$$E_t \left[\sup_{s \in [t, T]} |\bar{\psi}(\Delta \mathcal{X}_s)|^q \right] \leq E_t \left[\sup_{s \in [t, T]} |\Delta \mathcal{X}_s|^2 \right] < \infty. \quad (6.27)$$

Thus, $\underline{\mathcal{L}}^{\pm}, \overline{\mathcal{L}}^{\pm} \in \mathbb{C}_{\mathbf{F}^t}^q([t, T])$ by (2.16), and it follows that $\eta_{\pm} \in \mathbb{L}^q(\mathcal{F}_T^t)$. Then Theorem 4.1 of [15] shows that the DRBSDE($P_0^t, \eta_{\pm}, \mathbf{f}_{\pm}, \underline{\mathcal{L}}^{\pm}, \overline{\mathcal{L}}^{\pm}$) admits a unique solution $(\mathcal{Y}^{\pm}, \mathcal{Z}^{\pm}, \underline{\mathcal{X}}^{\pm}, \overline{\mathcal{X}}^{\pm}) \in \mathbb{G}_{\mathbf{F}^t}^q([t, T])$. For any $(s, \omega, y, z) \in [t, T] \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d$, (2.20), (2.5) and (2.21) imply that

$$\begin{aligned} \eta_-(\omega) &\leq h(\tilde{X}_T^{\Theta*}(\omega)) \leq \eta_+(\omega), \quad \mathbf{f}_-(s, \omega, y, z) \leq f_T^{\Theta*}(s, \omega, y, z) \leq \mathbf{f}_+(s, \omega, y, z), \\ \underline{\mathcal{L}}_s^-(\omega) &\leq \underline{\mathcal{L}}_s^{\Theta*}(\omega) \leq \underline{\mathcal{L}}_s^+(\omega), \quad \text{and} \quad \overline{\mathcal{L}}_s^-(\omega) \leq \overline{\mathcal{L}}_s^{\Theta*}(\omega) \leq \overline{\mathcal{L}}_s^+(\omega). \end{aligned}$$

Clearly, there inequalities also hold if Θ^* is replaced by Θ . Proposition 1.1 then yields that P_0^t -a.s.

$$\mathcal{Y}_s^- \leq \mathcal{Y}_s^{\Theta*}(T, h(\tilde{X}_T^{\Theta*})) \leq \mathcal{Y}_s^+, \quad \text{and} \quad \mathcal{Y}_s^- \leq \mathcal{Y}_s^{\Theta}(T, h(\tilde{X}_T^{\Theta})) \leq \mathcal{Y}_s^+, \quad \forall s \in [t, T]. \quad (6.28)$$

By (6.27), the processes $\widehat{\mathcal{Y}}_s^{\pm} \triangleq \mathcal{Y}_s^{\pm} \mp \gamma \bar{\psi}(\Delta \mathcal{X}_s)$, $s \in [t, T]$ are of $\mathbb{C}_{\mathbf{F}^t}^q([t, T])$. Let

$$\begin{aligned} \Delta \tilde{b}_s &\triangleq b(r, \tilde{X}_s^{\Theta*}, \mu'_s, \nu_s) - b(r, \tilde{X}_s^{\Theta}, \mu_s, \nu_s), \quad \Delta \tilde{\sigma}_s \triangleq \sigma(r, \tilde{X}_s^{\Theta*}, \mu'_s, \nu_s) - \sigma(r, \tilde{X}_s^{\Theta}, \mu_s, \nu_s) \\ \text{and} \quad \widehat{\mathcal{Z}}_s^{\pm} &\triangleq (\mathcal{Z}_s^{\pm} \mp \gamma ((D\bar{\psi})(\Delta \mathcal{X}_s))^T \Delta \tilde{\sigma}_s), \quad \forall s \in [t, T]. \end{aligned}$$

Similar to (6.22), Itô's formula implies that $(\widehat{\mathcal{Y}}^{\pm}, \widehat{\mathcal{Z}}^{\pm}, \underline{\mathcal{X}}^{\pm}, \overline{\mathcal{X}}^{\pm})$ solves the DRBSDE($P_0^t, \eta_{\pm}, \hat{\mathbf{f}}^{\pm}, \underline{\mathcal{L}}^{\Theta}, \overline{\mathcal{L}}^{\Theta}$) with

$$\begin{aligned} \hat{\mathbf{f}}^{\pm}(s, \omega, y, z) &\triangleq \mathbf{f}_{\pm}(s, \omega, y \pm \gamma \bar{\psi}(\Delta \mathcal{X}_s(\omega)), z \pm \gamma ((D\bar{\psi})(\Delta \mathcal{X}_s(\omega)))^T \Delta \tilde{\sigma}_s(\omega)) \pm \gamma ((D\bar{\psi})(\Delta \mathcal{X}_r(\omega)))^T \Delta \tilde{b}_r(\omega) \\ &\quad \pm \frac{1}{2} \gamma \text{trace} \left(D^2 \bar{\psi}(\Delta \mathcal{X}_r(\omega)) \Delta \tilde{\sigma}_r(\omega) (\Delta \tilde{\sigma}_r(\omega))^T \right), \quad \forall (s, \omega, y, z) \in [t, T] \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d. \end{aligned}$$

The measurability of $(\mathbf{f}_{\pm}, b, \sigma, \tilde{X}^{\Theta*}, \tilde{X}^{\Theta}, \mu', \mu, \nu)$ and the Lipschitz continuity of \mathbf{f}_{\pm} in (y, z) imply that $\hat{\mathbf{f}}^{\pm}$ are also $\mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) / \mathcal{B}(\mathbb{R})$ -measurable functions that are Lipschitz continuous in (y, z) with coefficient γ . Then we can deduce from (6.25), (2.2), (2.10) and (6.26) that

$$\begin{aligned} |\hat{\mathbf{f}}^{\pm}(s, 0, 0)| &\leq |\mathbf{f}_{\pm}(s, 0, 0)| + \gamma^2 \bar{\psi}(\Delta \mathcal{X}_s) + |((D\bar{\psi})(\Delta \mathcal{X}_s))| (\gamma^2 |\Delta \tilde{\sigma}_s| + \gamma |\Delta \tilde{b}_s|) + \frac{1}{2} \gamma \text{trace} \left(D^2 \bar{\psi}(\Delta \mathcal{X}_r) \Delta \tilde{\sigma}_r (\Delta \tilde{\sigma}_r)^T \right) \\ &\leq |\mathbf{f}_{\pm}(s, 0, 0)| + \gamma^2 |\Delta \mathcal{X}_s|^{2/q} + c_0 \left(\kappa_{\psi} + |\Delta \mathcal{X}_s|^{\frac{2}{q}-1} \right) \left(|\Delta \mathcal{X}_s| + \rho_{\mathbb{U}}^{\lambda}(\mu'_s, \mu_s) \right) \\ &\quad + \frac{1}{2} \gamma \left(\mathbf{1}_{\{|\Delta \mathcal{X}_s|=0\}} + \mathbf{1}_{\{|\Delta \mathcal{X}_s|>0\}} \kappa_{\psi} (1 \wedge |\Delta \mathcal{X}_s|^{\frac{2}{q}-2}) \right) \left(|\Delta \mathcal{X}_s|^2 + \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s) \right) \\ &\leq |\mathbf{f}_{\pm}(s, 0, 0)| + c_0 \kappa_{\psi} \left(|\Delta \mathcal{X}_s| + \rho_{\mathbb{U}}^{\lambda}(\mu'_s, \mu_s) + |\Delta \mathcal{X}_s|^{2/q} + \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s) \right), \quad s \in [t, T], \end{aligned} \quad (6.29)$$

where we used the Young's inequality in the last step:

$$|\Delta \mathcal{X}_s|^{\frac{2}{q}-1} \rho_{\mathbb{U}}^{\lambda}(\mu'_s, \mu_s) \leq c_0 \left(|\Delta \mathcal{X}_s|^{2/q} + \rho_{\mathbb{U}}^{\frac{2\lambda}{q}}(\mu'_s, \mu_s) \right) \leq c_0 \left(|\Delta \mathcal{X}_s|^{2/q} + \rho_{\mathbb{U}}^{\lambda}(\mu'_s, \mu_s) + \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s) \right).$$

Similarly, one can deduce from (6.25) (2.2), (2.10), Hölder's inequality and (2.7) that

$$\begin{aligned} E_t \left[\left(\int_t^T |((D\bar{\psi})(\Delta \mathcal{X}_s))^T \Delta \tilde{\sigma}_s|^2 ds \right)^{q/2} \right] &\leq c_0 E_t \left[\left(\int_t^T (\kappa_{\psi}^2 + |\Delta \mathcal{X}_s|^{\frac{4}{q}-2}) (|\Delta \mathcal{X}_s|^2 + \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s)) ds \right)^{q/2} \right] \\ &\leq c_0 E_t \left[\left(\kappa_{\psi}^q + \sup_{s \in [t, T]} |\Delta \mathcal{X}_s|^{2-q} \right) \left(\sup_{s \in [t, T]} |\Delta \mathcal{X}_s|^q + \left(\int_t^T \rho_{\mathbb{U}}^2(\mu'_s, \mu_s) ds \right)^{\frac{\lambda q}{2}} \right) \right] \\ &\leq c_0 \kappa_{\psi}^q \left(\left\{ E_t \left[\sup_{s \in [t, T]} |\Delta \mathcal{X}_s|^2 \right] \right\}^{q/2} + \left\{ E_t \int_t^T \rho_{\mathbb{U}}^2(\mu'_s, \mu_s) ds \right\}^{\frac{\lambda q}{2}} \right) + c_0 E_t \left[1 + \sup_{s \in [t, T]} |\Delta \mathcal{X}_s|^2 + \int_t^T \rho_{\mathbb{U}}^2(\mu'_s, \mu_s) ds \right] < \infty, \end{aligned}$$

where we used the Young's inequality in the last step:

$$\sup_{s \in [t, T]} |\Delta \mathcal{X}_s|^{2-q} \left(\int_t^T \rho_{\mathbb{U}}^2(\mu'_s, \mu_s) ds \right)^{\frac{\lambda q}{2}} \leq \frac{(1-\lambda)q}{2} + \frac{2-q}{2} \sup_{s \in [t, T]} |\Delta \mathcal{X}_s| + \frac{\lambda q}{2} \int_t^T \rho_{\mathbb{U}}^2(\mu'_s, \mu_s) ds.$$

Thus, $\widehat{\mathcal{X}}^{\pm} \in \mathbb{H}_{\mathbb{F}}^{2,q}([t, T], \mathbb{R}^d)$. Then Proposition 1.2 implies that

$$E_t \left[\sup_{s \in [t, T]} |\widehat{\mathcal{Y}}_s^+ - \widehat{\mathcal{Y}}_s^-|^{\varpi} \right] \leq c_{\varpi} \left\{ E_t[|\eta_+ - \eta_-|^{\varpi}] + E_t \left[\left(\int_t^T |\hat{f}^+(r, \widehat{\mathcal{Y}}_r^-, \widehat{\mathcal{Z}}_r^-) - \hat{f}^-(r, \widehat{\mathcal{Y}}_r^-, \widehat{\mathcal{Z}}_r^-)| dr \right)^{\varpi} \right] \right\}. \quad (6.30)$$

Similar to (6.29), the Lipschitz continuity of f_T^{Θ} , (6.25), (2.2), (2.10) and (6.26) imply that

$$\begin{aligned} |\hat{f}^+(s, \widehat{\mathcal{Y}}_s^-, \widehat{\mathcal{Z}}_s^-) - \hat{f}^-(s, \widehat{\mathcal{Y}}_s^-, \widehat{\mathcal{Z}}_s^-)| &\leq 2\gamma^2 \overline{\psi}(\Delta \mathcal{X}_s) + 2\gamma^2 \left| ((D\overline{\psi})(\Delta \mathcal{X}_s))^T \Delta \tilde{\sigma}_s \right| + 2\gamma |\Delta \mathcal{X}_s|^{2/q} \\ &\quad + 2\gamma \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s) + 2\gamma \left| ((D\overline{\psi})(\Delta \mathcal{X}_r))^T \Delta \tilde{b}_r \right| + \gamma \text{trace} \left(D^2 \overline{\psi}(\Delta \mathcal{X}_r) \Delta \tilde{\sigma}_r (\Delta \tilde{\sigma}_r)^T \right) \\ &\leq c_0 \kappa_{\psi} \left(|\Delta \mathcal{X}_s| + \rho_{\mathbb{U}}^{\lambda}(\mu'_s, \mu_s) + |\Delta \mathcal{X}_s|^{2/q} + \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s) \right), \quad s \in [t, T]. \end{aligned}$$

Putting this back into (6.30), we can deduce from (6.28), (2.11) and Hölder's inequality that

$$\begin{aligned} E_t \left[\sup_{s \in [t, T]} |Y_s^{\Theta^*}(T, h(\tilde{X}_T^{\Theta^*})) - Y_s^{\Theta}(T, h(\tilde{X}_T^{\Theta}))|^{\varpi} \right] &\leq E_t \left[\sup_{s \in [t, T]} |\mathcal{Y}_s^+ - \mathcal{Y}_s^-|^{\varpi} \right] \\ &\leq c_{\varpi} E_t \left[\sup_{s \in [t, T]} |\widehat{\mathcal{Y}}_s^+ - \widehat{\mathcal{Y}}_s^-|^{\varpi} \right] + c_{\varpi} E_t \left[\sup_{s \in [t, T]} |\overline{\psi}(\Delta \mathcal{X}_s)|^{\varpi} \right] \\ &\leq c_{\varpi} \kappa_{\psi}^{\varpi} \left\{ E_t \left[\sup_{s \in [t, T]} |\Delta \mathcal{X}_s|^{\varpi} \right] + E_t \left[\sup_{s \in [t, T]} |\Delta \mathcal{X}_s|^{\frac{2\varpi}{q}} \right] + E_t \left[\left(\int_t^T \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s) ds \right)^{\frac{\varpi}{2}} \right] + E_t \left[\left(\int_t^T \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s) ds \right)^{\varpi} \right] \right\} \\ &\leq c_{\varpi} \kappa_{\psi}^{\varpi} \left\{ E_t \left[\left(\int_t^T \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s) ds \right)^{\frac{\varpi}{2}} \right] + E_t \left[\left(\int_t^T \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s) ds \right)^{\frac{\varpi}{q}} \right] + E_t \left[\left(\int_t^T \rho_{\mathbb{U}}^{2\lambda}(\mu'_s, \mu_s) ds \right)^{\varpi} \right] \right\}. \end{aligned}$$

Hence, (2.22) holds. The proof of (2.24) is similar. \square

6.2 Proof of Dynamic Programming Principle

Lemma 6.3. *Let $0 \leq t \leq s \leq T$. For any $\omega \in \Omega^t$ and $\delta > 0$, $O_{\delta}^s(\omega) \triangleq \left\{ \omega' \in \Omega^t : \sup_{r \in [t, s]} |\omega'(r) - \omega(r)| < \delta \right\}$ is a \mathcal{F}_s^t -measurable open subset of Ω^t .*

Proof: Let $\omega \in \Omega^t$ and $\delta > 0$. Given $\omega' \in O_{\delta}^s(\omega)$, for any $\omega'' \in O_{\delta'}(\omega')$ with $\delta' \triangleq \delta - \sup_{r \in [t, s]} |\omega'(r) - \omega(r)|$, one has

$$\sup_{r \in [t, s]} |\omega''(r) - \omega(r)| \leq \|\omega'' - \omega'\|_t + \sup_{r \in [t, s]} |\omega'(r) - \omega(r)| < \delta' + \sup_{r \in [t, s]} |\omega'(r) - \omega(r)| = \delta.$$

Thus $O_{\delta'}(\omega') \subset O_{\delta}^s(\omega)$, which shows that $O_{\delta}^s(\omega)$ is an open subset of Ω^t . Moreover, since Ω^t is the set of \mathbb{R}^d -valued continuous functions on $[t, T]$ starting from 0, we see that

$$O_{\delta}^s(\omega) = \bigcap_{r \in \mathbb{Q}_{t, s}} \left\{ \omega' \in \Omega^t : |\omega'(r) - \omega(r)| < \delta \right\} = \bigcap_{r \in \mathbb{Q}_{t, s}} \left\{ \omega' \in \Omega^t : B_r^t(\omega') \in O_{\delta}(\omega(r)) \right\} \in \mathcal{F}_s^t. \quad \square$$

Given $t \in [0, T]$ and $\beta \in \mathfrak{B}^t$, we define

$$I(t, x, \beta) \triangleq \sup_{\mu \in \mathcal{U}^t} \tilde{Y}_t^{t, x, \mu, \beta(\mu)} \left(T, h(\tilde{X}_T^{t, x, \mu, \beta(\mu)}) \right), \quad \forall x \in \mathbb{R}^k.$$

Taking supremum over $\mu \in \mathcal{U}^t$ in (2.18) and (2.28) implies that

$$\underline{l}(t, x) \leq I(t, x, \beta) \leq \bar{l}(t, x), \quad \forall x \in \mathbb{R}^k, \quad (6.31)$$

and that

$$\text{the function } x \rightarrow I(t, x, \beta) \text{ is continuous.} \quad (6.32)$$

Similarly, for any $\alpha \in \mathcal{A}^t$, $I(t, x, \alpha) \triangleq \sup_{\nu \in \mathcal{V}^t} \tilde{Y}_t^{t, x, \alpha \langle \nu \rangle, \nu} \left(T, h(\tilde{X}_T^{t, x, \alpha \langle \nu \rangle, \nu}) \right) \in (\underline{l}(t, x), \bar{l}(t, x))$ is continuous in $x \in \mathbb{R}^k$.

Proof of Theorem 2.1: Let $\{t_n\}_{n \in \mathbb{N}}$ denote the countable set $\mathbb{Q}_{t, T}$ and let $\{\tau_{\mu, \beta} : \mu \in \mathcal{U}^t, \beta \in \mathfrak{B}^t\}$ be a family of $\mathbb{Q}_{t, T}$ -valued \mathbf{F}^t -stopping times.

1) We fix $\varepsilon > 0$. For any $n \in \mathbb{N}$ and $x \in \mathbb{R}^k$, since $w_1(t_n, x)$ is finite by (2.27), there exists a $\beta_x^n \in \mathfrak{B}^{t_n}$ such that

$$w_1(t_n, x) \leq I(t_n, x, \beta_x^n) \leq w_1(t_n, x) + \frac{1}{3}\varepsilon. \quad (6.33)$$

The continuity of $I(t_n, \cdot, \beta_x^n)$ by (6.32) and Proposition 2.1 assure that there exists a $\delta_n(x) > 0$ such that

$$|I(t_n, x', \beta_x^n) - I(t_n, x, \beta_x^n)| < \frac{1}{3}\varepsilon, \quad \text{and} \quad |w_1(t_n, x') - w_1(t_n, x)| < \frac{1}{3}\varepsilon, \quad \forall x' \in O_{\delta_n(x)}(x). \quad (6.34)$$

Given $n \in \mathbb{N}$, Lindelöf's covering theorem (see e.g. Theorem VIII.6.3 of [14]) shows that there exists a sequence $\{x_i^n\}_{i \in \mathbb{N}}$ of \mathbb{R}^k satisfying $\bigcup_{i \in \mathbb{N}} \tilde{O}_i^n = \mathbb{R}^k$ with $\tilde{O}_i^n \triangleq O_{\delta_n(x_i^n)}(x_i^n)$. For any $i \in \mathbb{N}$, let $\beta_i^n \triangleq \beta_{x_i^n}^n$. We can deduce from (6.33) and (6.34) that for any $x' \in \tilde{O}_i^n$

$$I(t_n, x', \beta_i^n) < I(t_n, x_i^n, \beta_i^n) + \frac{1}{3}\varepsilon \leq w_1(t_n, x_i^n) + \frac{2}{3}\varepsilon < w_1(t_n, x') + \varepsilon. \quad (6.35)$$

Now, fix $(\beta, \mu) \in \mathfrak{B}^t \times \mathcal{U}^t$. We simply denote $\tau_{\mu, \beta}$ by τ and set $\Theta = (t, x, \mu, \beta \langle \mu \rangle)$. For any $n, i \in \mathbb{N}$, define

$$A_i^n \triangleq \{\tau = t_n\} \cap \{\tilde{X}_{t_n}^\Theta \in \tilde{O}_i^n \setminus \bigcup_{j < i} \tilde{O}_j^n\} \in \mathcal{F}_{t_n}^t \cap \mathcal{F}_\tau^t.$$

Let $m \in \mathbb{N}$ and $A_m \triangleq \bigcup_{n, i=1}^m A_i^n \in \mathcal{F}_\tau^t$. Proposition 4.10 shows that

$$\beta^m(r, \omega, u) \triangleq \begin{cases} \beta_i^n(r, \Pi_{t, t_n}(\omega), u), & \text{if } (r, \omega) \in [\tau, T]_{A_i^n} = [t_n, T] \times A_i^n \text{ for } n, i \in \{1 \cdots, m\}, \\ \beta(r, \omega, u), & \text{if } (r, \omega) \in [t, \tau] \cup [\tau, T]_{A_m} \end{cases} \quad \forall u \in \mathbb{U}$$

defines a \mathfrak{B}^t -strategy such that it holds for P_0^t -a.s. $\omega \in \Omega^t$ that for any $r \in [\tau(\omega), T]$

$$(\beta^m \langle \mu \rangle)_r^{\tau, \omega} = \begin{cases} (\beta_i^n \langle \mu^{t_n, \omega} \rangle)_r, & \text{if } \omega \in A_i^n \text{ for } n, i \in \{1 \cdots, m\}, \\ (\beta \langle \mu \rangle)_r^{\tau, \omega}, & \text{if } \omega \in A_m. \end{cases} \quad (6.36)$$

Let $\Theta_m \triangleq (t, x, \mu, \beta^m \langle \mu \rangle)$. As $\beta^m \langle \mu \rangle = \beta \langle \mu \rangle$ on $[\tau, \tau]$, we see from (2.15) that P_0^t -a.s.

$$\tilde{X}_s^{\Theta_m} = X_s^{\Theta_m} = X_s^\Theta = \tilde{X}_s^\Theta, \quad \forall s \in [t, \tau]. \quad (6.37)$$

Hence, for any \mathcal{F}_τ^t -measurable random variable ξ with $\underline{L}_\tau^{\Theta_m} = \underline{L}_\tau^\Theta \leq \xi \leq \bar{L}_\tau^{\Theta_m} = \bar{L}_\tau^\Theta$, P_0^t -a.s., the DRBSDE($P^t, \xi, f_\tau^{\Theta_m}, \underline{L}_{\tau \wedge \cdot}^{\Theta_m}, \bar{L}_{\tau \wedge \cdot}^{\Theta_m}$) and the DRBSDE($P^t, \xi, f_\tau^\Theta, \underline{L}_{\tau \wedge \cdot}^\Theta, \bar{L}_{\tau \wedge \cdot}^\Theta$) are essentially the same. To wit, we have

$$(Y^{\Theta_m}(\tau, \xi), Z^{\Theta_m}(\tau, \xi), \underline{K}^{\Theta_m}(\tau, \xi), \bar{K}^{\Theta_m}(\tau, \xi)) = (Y^\Theta(\tau, \xi), Z^\Theta(\tau, \xi), \underline{K}^\Theta(\tau, \xi), \bar{K}^\Theta(\tau, \xi)). \quad (6.38)$$

Since $\tilde{X}_\tau^{\Theta_m} = \tilde{X}_\tau^\Theta$, P_0^t -a.s. by (6.37), it holds for P_0^t -a.s. $\omega \in \Omega^t$ that

$$(\Theta_m)_\tau^\omega = \left(\tau(\omega), \tilde{X}_{\tau(\omega)}^\Theta(\omega), \mu^{\tau, \omega}, (\beta^m \langle \mu \rangle)_\tau^{\tau, \omega} \right).$$

Then applying Proposition 4.8 and Proposition 4.7 with $\Theta = \Theta_m$ and $\xi = h(\tilde{X}_T^{\Theta_m})$, we can deduce from (6.36), Proposition 4.6 (1), (2.18) and (6.35) that

$$\begin{aligned} \left(\tilde{Y}_\tau^{\Theta_m}(T, h(\tilde{X}_T^{\Theta_m})) \right)(\omega) &= \tilde{Y}_{\tau(\omega)}^{(\Theta_m)_\tau^\omega} \left(T, h \left((\tilde{X}^{\Theta_m})_\tau^{\tau, \omega} \right) \right) = \tilde{Y}_{\tau(\omega)}^{(\Theta_m)_\tau^\omega} \left(T, h \left(\tilde{X}_T^{(\Theta_m)_\tau^\omega} \right) \right) \\ &= \tilde{Y}_{\tau(\omega)}^{\tau(\omega), \tilde{X}_{\tau(\omega)}^{\Theta_m}(\omega), \mu^{\tau, \omega}, (\beta(\mu))^{\tau, \omega}} \left(T, h \left(\tilde{X}_T^{\tau(\omega), \tilde{X}_{\tau(\omega)}^{\Theta_m}(\omega), \mu^{\tau, \omega}, (\beta(\mu))^{\tau, \omega}} \right) \right) \\ &\leq \mathbf{1}_{\{\omega \in A_m\}} \bar{l}(\tau(\omega), \tilde{X}_{\tau(\omega)}^{\Theta_m}(\omega)) + \sum_{n,i=1}^m \mathbf{1}_{\{\omega \in A_i^n\}} \tilde{Y}_{t_n}^{t_n, \tilde{X}_{t_n}^{\Theta_m}(\omega), \mu^{t_n, \omega}, \beta_i^n \langle \mu^{t_n, \omega} \rangle} \left(T, h \left(\tilde{X}_T^{t_n, \tilde{X}_{t_n}^{\Theta_m}(\omega), \mu^{t_n, \omega}, \beta_i^n \langle \mu^{t_n, \omega} \rangle} \right) \right) \end{aligned} \quad (6.39)$$

$$\leq \mathbf{1}_{\{\omega \in A_m\}} \bar{l}(\tau(\omega), \tilde{X}_{\tau(\omega)}^{\Theta_m}(\omega)) + \sum_{n,i=1}^m \mathbf{1}_{\{\omega \in A_i^n\}} I(t_n, \tilde{X}_{t_n}^{\Theta_m}(\omega), \beta_i^n) \leq \xi_m(\omega), \quad \text{for } P_0^t\text{-a.s. } \omega \in \Omega^t, \quad (6.40)$$

where $\xi_m \triangleq \mathbf{1}_{A_m} \bar{l}(\tau, \tilde{X}_\tau^{\Theta_m}) + \mathbf{1}_{A_m^c} \left((w_1(\tau, \tilde{X}_\tau^{\Theta_m}) + \varepsilon) \wedge \bar{l}(\tau, \tilde{X}_\tau^{\Theta_m}) \right)$.

The continuity of \bar{l} and (2.29) show that $\bar{l}(\tau, \tilde{X}_\tau^{\Theta_m}), w_1(\tau, \tilde{X}_\tau^{\Theta_m}) \in \mathcal{F}_\tau^t$, so is ξ_m . Also, we see from (2.27) that

$$\underline{L}_\tau^{\Theta_m} = \underline{l}(\tau, \tilde{X}_\tau^{\Theta_m}) \leq w_1(\tau, \tilde{X}_\tau^{\Theta_m}) \leq \bar{l}(\tau, \tilde{X}_\tau^{\Theta_m}) = \bar{L}_\tau^{\Theta_m} \quad \text{and} \quad \underline{L}_\tau^{\Theta_m} = \underline{l}(\tau, \tilde{X}_\tau^{\Theta_m}) \leq \xi_m \leq \bar{l}(\tau, \tilde{X}_\tau^{\Theta_m}) = \bar{L}_\tau^{\Theta_m}, \quad P_0^t\text{-a.s.}$$

Thus both $Y^{\Theta_m}(\tau, \xi_m)$ and $Y^{\Theta_m}(\tau, w_1(\tau, \tilde{X}_\tau^{\Theta_m}))$ is well-posed. Then Proposition 1.2 implies that

$$|\tilde{Y}_t^{\Theta_m}(\tau, \xi_m) - \tilde{Y}_t^{\Theta_m}(\tau, w_1(\tau, \tilde{X}_\tau^{\Theta_m}))| \leq c_0 \|\xi_m - w_1(\tau, \tilde{X}_\tau^{\Theta_m})\|_{\mathbb{L}^q(\mathcal{F}_\tau^t)} \leq c_0 \|\mathbf{1}_{A_m}(\bar{l}(\tau, \tilde{X}_\tau^{\Theta_m}) - \underline{l}(\tau, \tilde{X}_\tau^{\Theta_m}))\|_{\mathbb{L}^q(\mathcal{F}_\tau^t)} + c_0 \varepsilon. \quad (6.41)$$

Applying (2.17) with $(\zeta, \tau, \xi) = (\tau, T, h(\tilde{X}_T^{\Theta_m}))$, applying Proposition 1.1 to (6.40), and using (6.38) for $\xi = \xi_m$ yield that

$$\begin{aligned} \tilde{Y}_t^{\Theta_m}(T, h(\tilde{X}_T^{\Theta_m})) &= \tilde{Y}_t^{\Theta_m}(\tau, \tilde{Y}_\tau^{\Theta_m}(T, h(\tilde{X}_T^{\Theta_m}))) \leq \tilde{Y}_t^{\Theta_m}(\tau, \xi_m) = \tilde{Y}_t^{\Theta_m}(\tau, \xi_m) \\ &\leq \tilde{Y}_t^{\Theta_m}(\tau, w_1(\tau, \tilde{X}_\tau^{\Theta_m})) + c_0 \|\mathbf{1}_{A_m}(\bar{l}(\tau, \tilde{X}_\tau^{\Theta_m}) - \underline{l}(\tau, \tilde{X}_\tau^{\Theta_m}))\|_{\mathbb{L}^q(\mathcal{F}_\tau^t)} + c_0 \varepsilon. \end{aligned} \quad (6.42)$$

Since $\bigcup_{n,i \in \mathbb{N}} A_i^n = \Omega^t$, one has $\lim_{m \rightarrow \infty} \downarrow A_m = \emptyset$. By (2.3) and Hölder's inequality,

$$\|\bar{l}(\tau, \tilde{X}_\tau^{\Theta_m}) - \underline{l}(\tau, \tilde{X}_\tau^{\Theta_m})\|_{\mathbb{L}^q(\mathcal{F}_\tau^t)} \leq \sup_{s \in [t, T]} |\bar{l}(s, 0)| + \sup_{s \in [t, T]} |\underline{l}(s, 0)| + 2\gamma \|\tilde{X}^{\Theta_m}\|_{\mathbb{C}_{\mathbb{F}^t}^2([t, T], \mathbb{R}^k)} < \infty. \quad (6.43)$$

Hence, the Dominated Convergence Theorem shows that $\lim_{m \rightarrow \infty} \downarrow \|\mathbf{1}_{A_m}(\bar{l}(\tau, \tilde{X}_\tau^{\Theta_m}) - \underline{l}(\tau, \tilde{X}_\tau^{\Theta_m}))\|_{\mathbb{L}^q(\mathcal{F}_\tau^t)} = 0$. Letting $m \rightarrow \infty$ in (6.42) gives that

$$\tilde{Y}_t^{t, x, \mu, \beta^m \langle \mu \rangle}(T, h(X_T^{t, x, \mu, \beta^m \langle \mu \rangle})) \leq \tilde{Y}_t^{t, x, \mu, \beta \langle \mu \rangle}(\tau_{\mu, \beta}, w_1(\tau_{\mu, \beta}, X_{\tau_{\mu, \beta}}^{t, x, \mu, \beta \langle \mu \rangle})) + c_0 \varepsilon.$$

Taking supremum over $\mu \in \mathcal{U}^t$, we obtain

$$w_1(t, x) \leq I(t, x, \beta^m) \leq \sup_{\mu \in \mathcal{U}^t} \tilde{Y}_t^{t, x, \mu, \beta \langle \mu \rangle}(\tau_{\mu, \beta}, w_1(\tau_{\mu, \beta}, X_{\tau_{\mu, \beta}}^{t, x, \mu, \beta \langle \mu \rangle})) + c_0 \varepsilon.$$

Then taking infimum over $\beta \in \mathfrak{B}^t$ and letting $\varepsilon \rightarrow 0$ yield (2.30). Similarly, one has (2.31).

2) Next, assume (\mathbf{V}_λ) for some $\lambda \in (0, 1)$. We shall show the inverse of (2.31).

a) Fix $(\beta, \mu) \in \hat{\mathfrak{B}}^t \times \mathcal{U}^t$. We simply denote $\tau_{\mu, \beta}$ by τ . For some $\kappa > 0$ and some non-negative measurable process Ψ on $(\Omega^t, \mathcal{F}_T^t)$ with $C_\Psi \triangleq E_t \int_t^T \Psi_r^2 dr < \infty$, it holds $dr \times dP_0^t$ -a.s. that

$$[\beta(r, \omega, u)]_{\mathbb{V}} \leq \Psi_r(\omega) + \kappa[u]_{\mathbb{U}}, \quad \forall u \in \mathbb{U}.$$

Similar to (6.104), applying Proposition 4.3 with $\xi = \int_\tau^T \Psi_r^2 dr \in \mathbb{L}^1(\mathcal{F}_T^t)$, we can deduce that for all $\omega \in \Omega^t$ except on a P_0^t -null set \mathcal{N}_0

$$E_{\tau(\omega)} \left[\int_{\tau(\omega)}^T (\Psi_r^{\tau, \omega})^2 dr \right] = E_t \left[\int_\tau^T \Psi_r^2 dr \middle| \mathcal{F}_\tau^t \right](\omega) \leq E_t \left[\int_t^T \Psi_r^2 dr \middle| \mathcal{F}_\tau^t \right](\omega) < \infty. \quad (6.44)$$

For any $n \in \mathbb{N}$, similar to (6.105) and the conclusion that follows, it holds for all $\omega \in \Omega^t$ except on a P_0^t -a.s. null set \mathcal{N}_n^1 that for $dr \times dP_0^{t_n}$ -a.s. $(r, \tilde{\omega}) \in [t_n, T] \times \Omega^{t_n}$

$$[\beta^{t_n, \omega}(r, \tilde{\omega}, u)]_{\mathbb{V}} \leq \Psi_r^{t_n, \omega}(\tilde{\omega}) + \kappa[u]_{\mathbb{U}}, \quad \forall u \in \mathbb{U}. \quad (6.45)$$

Also, Proposition 4.6 (2) shows that there exists another P_0^t -null set \mathcal{N}_n^2 such that $\beta^{t_n, \omega} \in \widehat{\mathfrak{B}}^{t_n}$ for any $\omega \in (\mathcal{N}_n^2)^c$.

Now, let us set $\varpi \triangleq \frac{\lambda+1}{2\lambda} \wedge q > 1$, clearly, $\lambda\varpi \leq \frac{\lambda+1}{2} < 1$. Also, we fix $\varepsilon \in \left(0, \frac{1}{4} \wedge (4C_\Psi)^{\frac{2\lambda\varpi}{\lambda\varpi-1}}\right)$. For $G_\varepsilon^1 \triangleq \left\{\omega \in \Omega^t : E_t \left[\int_t^T \Psi_r^2 dr \middle| \mathcal{F}_\tau^t \right](\omega) > \varepsilon^{\frac{\lambda\varpi-1}{2\lambda\varpi}} \right\} \in \mathcal{F}_\tau^t$, one can deduce that

$$P_0^t(G_\varepsilon^1) \leq \varepsilon^{\frac{1-\lambda\varpi}{2\lambda\varpi}} E_t \left[\mathbf{1}_{G_\varepsilon^1} E_t \left[\int_t^T \Psi_r^2 dr \middle| \mathcal{F}_\tau^t \right] \right] \leq \varepsilon^{\frac{1-\lambda\varpi}{2\lambda\varpi}} E_t \int_t^T \Psi_r^2 dr = C_\Psi \varepsilon^{\frac{1-\lambda\varpi}{2\lambda\varpi}} < \frac{1}{4}. \quad (6.46)$$

Let $\mathfrak{k}(\varepsilon) \triangleq \lceil 1 - \log_2 \varepsilon \rceil$. Given $\mathfrak{k} \in \mathbb{N}$ with $\mathfrak{k} \geq \mathfrak{k}(\varepsilon)$, There exist a $\delta(\mathfrak{k}) > 0$ and a closed subset $F_{\mathfrak{k}}$ of Ω^t with $P_0^t(F_{\mathfrak{k}}) > 1 - 2^{-2\mathfrak{k}}$ such that for any $\omega, \omega' \in F_{\mathfrak{k}}$ with $\|\omega - \omega'\|_t < \delta(\mathfrak{k})$

$$\sup_{r \in [t, T]} \sup_{u \in \mathbb{U}} \rho_{\mathbb{V}}(\beta(r, \omega, u), \beta(r, \omega', u)) < 2^{-2\mathfrak{k}}. \quad (6.47)$$

As $G_{\mathfrak{k}} \triangleq \left\{\omega \in \Omega^t : E_t \left[\mathbf{1}_{F_{\mathfrak{k}}} \middle| \mathcal{F}_\tau^t \right](\omega) > 2^{-\mathfrak{k}} \right\} \in \mathcal{F}_\tau^t$, similar to (6.46), one can deduce that $P_0^t(G_{\mathfrak{k}}) \leq 2^{\mathfrak{k}} P_0^t(F_{\mathfrak{k}}^c) \leq 2^{-\mathfrak{k}}$.

Thus for $G_\varepsilon^2 \triangleq \bigcup_{\mathfrak{k} \geq \mathfrak{k}(\varepsilon)} G_{\mathfrak{k}}$, we have

$$P_0^t(G_\varepsilon^2) \leq 2^{1-\mathfrak{k}(\varepsilon)} \leq \varepsilon.$$

Using (4.2), (6.99) and applying Proposition 4.3 with $\xi = \mathbf{1}_{F_{\mathfrak{k}}^c}$, we obtain that for all $\omega \in \Omega^t$ except on a P_0^t -null set $\mathcal{N}_{\mathfrak{k}}$

$$P_0^{\tau(\omega)}((F_{\mathfrak{k}}^{\tau, \omega})^c) = P_0^{\tau(\omega)}((F_{\mathfrak{k}}^c)^{\tau, \omega}) = E_{\tau(\omega)}[\mathbf{1}_{(F_{\mathfrak{k}}^c)^{\tau, \omega}}] = E_{\tau(\omega)}[(\mathbf{1}_{F_{\mathfrak{k}}^c})^{\tau, \omega}] = E_t[\mathbf{1}_{F_{\mathfrak{k}}^c} \middle| \mathcal{F}_\tau^t](\omega). \quad (6.48)$$

b) As $\mathcal{F}_T^t = \mathcal{B}(\Omega^t)$ by (1.4), there exists an open set \tilde{O}_ε of Ω^t that includes

$$\tilde{G}_\varepsilon \triangleq G_\varepsilon^1 \cup G_\varepsilon^2 \cup \mathcal{N}_0 \cup \left(\bigcup_{n \in \mathbb{N}} \mathcal{N}_n^1 \right) \cup \left(\bigcup_{n \in \mathbb{N}} \mathcal{N}_n^2 \right) \cup \left(\bigcup_{\mathfrak{k} \geq \mathfrak{k}(\varepsilon)} \mathcal{N}_{\mathfrak{k}} \right),$$

and satisfies (see e.g. Proposition 15.11 of [40])

$$P_0^t(\tilde{O}_\varepsilon) < P_0^t(G_\varepsilon^1) + P_0^t(G_\varepsilon^2) + \varepsilon \leq C_\Psi \varepsilon^{\frac{1-\lambda\varpi}{2\lambda\varpi}} + 2\varepsilon < \frac{1}{4} + 2\varepsilon. \quad (6.49)$$

Fix $n \in \mathbb{N}$ such that $\{\tau = t_n\} \setminus \tilde{O}_\varepsilon \neq \emptyset$, we let $x \in \mathbb{R}^k$ and $\omega \in \{\tau = t_n\} \setminus \tilde{O}_\varepsilon$. Since $I(t_n, x, \beta^{t_n, \omega})$ is finite by (6.31), there exists a $\mu_{\omega, x}^n \in \mathcal{U}^{t_n}$ such that

$$\tilde{Y}_{t_n}^{t_n, x, \mu_{\omega, x}^n, \beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle} \left(T, h \left(\tilde{X}_T^{t_n, x, \mu_{\omega, x}^n, \beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle} \right) \right) \geq I(t_n, x, \beta^{t_n, \omega}) - \frac{1}{2}\varepsilon \geq \hat{w}_1(t_n, x) - \frac{1}{2}\varepsilon.$$

Let $\hat{\mathfrak{k}} = \mathfrak{k}_{\omega, x}^n \triangleq \mathfrak{k}(\varepsilon) \vee \left\lceil \frac{\lambda\varpi}{1-\lambda\varpi} \log_2 \left(\frac{8\kappa^2}{\varepsilon} E_{t_n} \int_{t_n}^T [(\mu_{\omega, x}^n)_r]_{\mathbb{U}}^2 dr \right) \right\rceil$. Given $x' \in O_\varepsilon(x)$ and $\omega' \in \left(\{\tau = t_n\} \cap O_{\delta(\hat{\mathfrak{k}})}^{t_n}(\omega) \right) \setminus \tilde{O}_\varepsilon$, applying (2.19) and (2.24) with $t = t_n$ and using Hölder's inequality, we obtain

$$\begin{aligned} & \left| \tilde{Y}_{t_n}^{t_n, x, \mu_{\omega, x}^n, \beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle} \left(T, h \left(\tilde{X}_T^{t_n, x, \mu_{\omega, x}^n, \beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle} \right) \right) - \tilde{Y}_{t_n}^{t_n, x', \mu_{\omega', x'}^n, \beta^{t_n, \omega'} \langle \mu_{\omega', x'}^n \rangle} \left(T, h \left(\tilde{X}_T^{t_n, x', \mu_{\omega', x'}^n, \beta^{t_n, \omega'} \langle \mu_{\omega', x'}^n \rangle} \right) \right) \right|^{\varpi} \\ & \leq c_0 |x - x'|^{\frac{2\varpi}{q}} + c_\lambda (\kappa_\psi)^{\varpi} E_{t_n} \left[\left(\int_{t_n}^T \rho_{\mathbb{V}}^2 \left((\beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle)_r, (\beta^{t_n, \omega'} \langle \mu_{\omega', x'}^n \rangle)_r \right) dr \right)^{\frac{\lambda\varpi}{2}} \right] \\ & \quad + c_\lambda (\kappa_\psi)^{\varpi} E_{t_n} \left[\left(\int_{t_n}^T \rho_{\mathbb{V}}^2 \left((\beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle)_r, (\beta^{t_n, \omega'} \langle \mu_{\omega', x'}^n \rangle)_r \right) dr \right)^{\lambda\varpi} \right]. \end{aligned} \quad (6.50)$$

For any $\tilde{\omega} \in F_{\hat{\mathfrak{t}}}^{t_n, \omega, \omega'} \triangleq F_{\hat{\mathfrak{t}}}^{t_n, \omega} \cap F_{\hat{\mathfrak{t}}}^{t_n, \omega'}$, since $\omega' \in O_{\delta(\hat{\mathfrak{t}})}^{t_n}(\omega)$, one has $\sup_{r \in [t, T]} |(\omega \otimes_{t_n} \tilde{\omega})(r) - (\omega' \otimes_{t_n} \tilde{\omega})(r)| = \sup_{r \in [t, t_n]} |\omega'(r) - \omega(r)| < \delta(\hat{\mathfrak{t}})$. It then follows from (6.47) that for any $r \in [t_n, T]$

$$\begin{aligned} \rho_{\mathbb{V}}\left((\beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle)_r, (\beta^{t_n, \omega'} \langle \mu_{\omega, x}^n \rangle)_r(\tilde{\omega})\right) &= \rho_{\mathbb{V}}\left(\beta(r, \omega \otimes_{t_n} \tilde{\omega}, (\mu_{\omega, x}^n)_r(\omega \otimes_{t_n} \tilde{\omega})), \beta(r, \omega_i^n \otimes_{t_n} \tilde{\omega}, (\mu_{\omega, x}^n)_r(\omega \otimes_{t_n} \tilde{\omega}))\right) \\ &< 2^{-2\hat{\mathfrak{t}}} < 2^{-\mathfrak{t}(\varepsilon)} < \varepsilon. \end{aligned} \quad (6.51)$$

Since both ω and ω_i^n belongs to $\{\tau = t_n\} \cap G_{\varepsilon}^c \cap G_{\hat{\mathfrak{t}}}^c \cap \mathcal{N}_{\hat{\mathfrak{t}}}^c \cap \mathcal{N}_0^c \cap (\mathcal{N}_n^1)^c \cap (\mathcal{N}_n^2)^c$, we see from (6.44) and (6.45) that

$$\begin{aligned} E_{t_n} \int_{t_n}^T \rho_{\mathbb{V}}^2\left((\beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle)_r, (\beta^{t_n, \omega'} \langle \mu_{\omega, x}^n \rangle)_r\right) dr \\ \leq 2 \int_{\Omega^{t_n}} \int_{t_n}^T \left(\left[\beta^{t_n, \omega}(r, \tilde{\omega}, (\mu_{\omega, x}^n)_r(\tilde{\omega})) \right]_{\mathbb{V}}^2 + \left[\beta^{t_n, \omega'}(r, \tilde{\omega}, (\mu_{\omega, x}^n)_r(\tilde{\omega})) \right]_{\mathbb{V}}^2 \right) dr dP_0^{t_n}(\tilde{\omega}) \\ \leq 4E_{t_n} \int_{t_n}^T (\Psi_r^{t_n, \omega})^2 dr + 4E_{t_n} \int_{t_n}^T (\Psi_r^{t_n, \omega'})^2 dr + 8\kappa^2 E_{t_n} \int_{t_n}^T [(\mu_{\omega, x}^n)_r]_{\mathbb{U}}^2 dr \\ \leq 4E_t \left[\int_t^T \Psi_r^2 dr \middle| \mathcal{F}_{\tau}^t \right](\omega) + 4E_t \left[\int_t^T \Psi_r^2 dr \middle| \mathcal{F}_{\tau}^t \right](\omega') + \varepsilon (2^{\hat{\mathfrak{t}}})^{\frac{1-\lambda\varpi}{\lambda\varpi}} \leq 8\varepsilon^{\frac{\lambda\varpi-1}{2\lambda\varpi}} + \varepsilon (2^{\hat{\mathfrak{t}}})^{\frac{1-\lambda\varpi}{\lambda\varpi}}, \end{aligned} \quad (6.52)$$

and (6.48) shows that

$$P_0^{t_n} \left((F_{\hat{\mathfrak{t}}}^{t_n, \omega, \omega'})^c \right) \leq P_0^{t_n} \left((F_{\hat{\mathfrak{t}}}^{t_n, \omega})^c \right) + P_0^{t_n} \left((F_{\hat{\mathfrak{t}}}^{t_n, \omega'})^c \right) = E_t [\mathbf{1}_{F_{\hat{\mathfrak{t}}}^c} | \mathcal{F}_{\tau}^t](\omega) + E_t [\mathbf{1}_{F_{\hat{\mathfrak{t}}}^c} | \mathcal{F}_{\tau}^t](\omega') \leq 2^{1-\hat{\mathfrak{t}}}.$$

Putting this together with (6.51) and (6.52) back into (6.50), one can deduce from Hölder's inequality that

$$\begin{aligned} &\left| \tilde{Y}_{t_n}^{t_n, x, \mu_{\omega, x}^n, \beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle} \left(T, h \left(\tilde{X}_T^{t_n, x, \mu_{\omega, x}^n, \beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle} \right) \right) - \tilde{Y}_{t_n}^{t_n, x', \mu_{\omega, x}^n, \beta^{t_n, \omega'} \langle \mu_{\omega, x}^n \rangle} \left(T, h \left(\tilde{X}_T^{t_n, x', \mu_{\omega, x}^n, \beta^{t_n, \omega'} \langle \mu_{\omega, x}^n \rangle} \right) \right) \right|^{\varpi} \\ &\leq c_0 \varepsilon^{\frac{2\varpi}{q}} + c_{\lambda}(\kappa_{\psi})^{\varpi} P_0^{t_n} \left(F_{\hat{\mathfrak{t}}}^{t_n, \omega, \omega'} \right) \varepsilon^{\lambda\varpi} \\ &\quad + c_{\lambda}(\kappa_{\psi})^{\varpi} (E_{t_n}[1])^{\frac{1}{2}} \left[P_0^{t_n} \left((F_{\hat{\mathfrak{t}}}^{t_n, \omega, \omega'})^c \right) \right]^{\frac{1-\lambda\varpi}{2}} \left(E_{t_n} \int_{t_n}^T \rho_{\mathbb{V}}^2 \left\{ (\beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle)_r, (\beta^{t_n, \omega'} \langle \mu_{\omega, x}^n \rangle)_r \right\} dr \right)^{\frac{\lambda\varpi}{2}} \\ &\quad + c_{\lambda}(\kappa_{\psi})^{\varpi} \left(P_0^{t_n} \left(F_{\hat{\mathfrak{t}}}^{t_n, \omega, \omega'} \right) \varepsilon^{2\lambda\varpi} + \left[P_0^{t_n} \left((F_{\hat{\mathfrak{t}}}^{t_n, \omega, \omega'})^c \right) \right]^{1-\lambda\varpi} \left(E_{t_n} \int_{t_n}^T \rho_{\mathbb{V}}^2 \left\{ (\beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle)_r, (\beta^{t_n, \omega'} \langle \mu_{\omega, x}^n \rangle)_r \right\} dr \right)^{\lambda\varpi} \right) \\ &\leq c_{\lambda}(\kappa_{\psi})^{\varpi} \varepsilon^{\lambda\varpi} + c_{\lambda}(\kappa_{\psi})^{\varpi} (\varepsilon^{\frac{1-\lambda\varpi}{2\lambda\varpi}} + \varepsilon)^{\frac{\lambda\varpi}{2}} + c_{\lambda}(\kappa_{\psi})^{\varpi} (\varepsilon^{\frac{1-\lambda\varpi}{2\lambda\varpi}} + \varepsilon)^{\lambda\varpi} \leq c_{\lambda}(\kappa_{\psi})^{\varpi} \varepsilon^{\frac{1-\lambda\varpi}{4}}, \end{aligned}$$

where We used $\lambda < 1 \leq 2/q$ and $2^{1-\hat{\mathfrak{t}}} \leq 2^{1-\mathfrak{t}(\varepsilon)} \leq \varepsilon$ in the second inequality. Moreover, Proposition 2.1 assures that there exists a $\lambda_n \in (0, \varepsilon)$ such that $|\hat{w}_1(t_n, x) - \hat{w}_1(t_n, x')| < \frac{1}{2}\varepsilon$ for any $x' \in O_{\lambda_n}(x)$. Therefore, it holds for any $x' \in O_{\lambda_n}(x)$ and $\omega' \in \left(\{\tau = t_n\} \cap O_{\delta(\hat{\mathfrak{t}})}^{t_n}(\omega) \right) \setminus \tilde{O}_{\varepsilon}$ that

$$\begin{aligned} \tilde{Y}_{t_n}^{t_n, x', \mu_{\omega, x}^n, \beta^{t_n, \omega'} \langle \mu_{\omega, x}^n \rangle} \left(T, h \left(\tilde{X}_T^{t_n, x', \mu_{\omega, x}^n, \beta^{t_n, \omega'} \langle \mu_{\omega, x}^n \rangle} \right) \right) &\geq \tilde{Y}_{t_n}^{t_n, x, \mu_{\omega, x}^n, \beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle} \left(T, h \left(\tilde{X}_T^{t_n, x, \mu_{\omega, x}^n, \beta^{t_n, \omega} \langle \mu_{\omega, x}^n \rangle} \right) \right) - c_0 \kappa_{\psi} \varepsilon^{\frac{1-\lambda\varpi}{4\varpi}} \\ &\geq \hat{w}_1(t_n, x) - \frac{1}{2}\varepsilon - c_0 \kappa_{\psi} \varepsilon^{\frac{1-\lambda\varpi}{4\varpi}} > \hat{w}_1(t_n, x') - \varepsilon - c_0 \kappa_{\psi} \varepsilon^{\frac{1-\lambda\varpi}{4\varpi}} > \hat{w}_1(t_n, x') - c_0 \kappa_{\psi} \varepsilon^{\frac{1-\lambda\varpi}{4\varpi}}. \end{aligned} \quad (6.53)$$

c) For any $n \in \mathbb{N}$, there exists a closed set F_{ε}^n of Ω^t that is included in $\{\tau = t_n\}$ and satisfies $P_0^t(F_{\varepsilon}^n) \geq (P_0^t\{\tau = t_n\} - \frac{\varepsilon}{2^n})^+$. Set $\mathfrak{N} \triangleq \{n \in \mathbb{N} : F_{\varepsilon}^n \setminus \tilde{O}_{\varepsilon} \neq \emptyset\}$. We claim that $\mathfrak{N} \neq \emptyset$: Assume not. Then $\bigcup_{n \in \mathbb{N}} F_{\varepsilon}^n \subset \tilde{O}_{\varepsilon}$ and it follows

$$\frac{3}{4} \geq \frac{1}{4} + 2\varepsilon > P_0^t(\tilde{O}_{\varepsilon}) \geq \sum_{n \in \mathbb{N}} P_0^t(F_{\varepsilon}^n) \geq \sum_{n \in \mathbb{N}} \left(P_0^t\{\tau = t_n\} - \frac{\varepsilon}{2^n} \right) = 1 - \varepsilon \geq \frac{3}{4}.$$

A contradiction appears.

Let $n \in \mathfrak{N}$. We see from Lemma 6.3 that $\left\{ \mathfrak{D}_{\omega,x}^n \triangleq O_{\delta(\mathfrak{k}_{\omega,x}^n)}^{t_n}(\omega) \times O_{\lambda_n}(x) : \omega \in F_\varepsilon^n \setminus \tilde{O}_\varepsilon, x \in \mathbb{R}^k \right\}$ together with $(\Omega^t \setminus (F_\varepsilon^n \setminus \tilde{O}_\varepsilon)) \times \mathbb{R}^k$ form an open cover of $\Omega^t \times \mathbb{R}^k$. Since the canonical space Ω^t is separable, the product space $\Omega^t \times \mathbb{R}^k$ is still separable and thus Lindelöf. Then one can find a sequence $\{(\omega_i^n, x_i^n)\}_{i \in \mathbb{N}}$ of $(F_\varepsilon^n \setminus \tilde{O}_\varepsilon) \times \mathbb{R}^k$ such that $(F_\varepsilon^n \setminus \tilde{O}_\varepsilon) \times \mathbb{R}^k \subset \bigcup_{i \in \mathbb{N}} \mathfrak{D}_i^n$. with $\mathfrak{D}_i^n \triangleq \mathfrak{D}_{\omega_i^n, x_i^n}^n$. We assume without loss of generality that

$$\tilde{\mathfrak{D}}_i^n \triangleq \mathfrak{D}_i^n \setminus \bigcup_{j < i} \mathfrak{D}_j^n \neq \emptyset, \quad \forall i \geq 2$$

and set $\tilde{\mathfrak{D}}_1^n \triangleq \mathfrak{D}_1^n$. Since $\{O_\delta^{t_n}(\omega) : \omega \in \Omega^t, \delta > 0\}$ are all $\mathcal{F}_{t_n}^t$ -measurable by Lemma 6.3 again, one can inductively show that each $\tilde{\mathfrak{D}}_i^n$ is a disjoint union of finitely many measurable rectangular sets $\{\mathfrak{A}_j^{n,i} \times \mathfrak{E}_j^{n,i} : \mathfrak{A}_j^{n,i} \in \mathcal{F}_{t_n}^t, \mathfrak{E}_j^{n,i} \in \mathcal{B}(\mathbb{R}^k)\}_{j=1}^{J_i^n}$. For any $i \in \mathbb{N}$, we set $(\mathfrak{k}_i^n, \beta_i^n, \mu_i^n) = (\mathfrak{k}_{\omega_i^n, x_i^n}^n, \beta_{t_n, \omega_i^n}^n, \mu_{\omega_i^n, x_i^n}^n)$.

d) We let $\Theta \triangleq (t, x, \mu, \beta\langle\mu\rangle)$ and define disjoint sets:

$$A_j^{n,i} \triangleq \{\tau = t_n\} \cap \mathfrak{A}_j^{n,i} \cap \{\tilde{X}_{t_n}^\Theta \in \mathfrak{E}_j^{n,i}\} \in \mathcal{F}_{t_n}^t \cap \mathcal{F}_\tau^t, \quad \forall n \in \mathfrak{N}, i \in \mathbb{N} \text{ and } j = 1, \dots, J_i^n.$$

Fix $m \in \mathbb{N}$ such that $\mathfrak{N}_m \triangleq \mathfrak{N} \cap \{1, \dots, m\} \neq \emptyset$. Setting $A_m \triangleq \Omega^t \setminus \left(\bigcup_{n \in \mathfrak{N}_m} \bigcup_{i=1}^m \bigcup_{j=1}^{J_i^n} A_j^{n,i} \right) \in \mathcal{F}_\tau^t$, we see from Proposition 4.9 that

$$\mu_r^m(\omega) \triangleq \begin{cases} (\mu_i^n)_r(\Pi_{t,t_n}(\omega)), & \text{if } (r, \omega) \in [\tau, T]_{A_j^{n,i}} = [t_n, T] \times A_j^{n,i} \text{ for } n \in \mathfrak{N}_m, i = 1 \dots, m \text{ and } j = 1 \dots, J_i^n, \\ \mu_r(\omega), & \text{if } (r, \omega) \in [t, \tau] \cup [\tau, T]_{A_m} \end{cases} \quad (6.54)$$

defines a \mathcal{U}^t -control such that for any $(r, \omega) \in [\tau, T]$

$$(\mu^m)_r^{\tau, \omega} = \begin{cases} (\mu_i^n)_r, & \text{if } \omega \in A_j^{n,i} \text{ for } n \in \mathfrak{N}_m, i = 1 \dots, m \text{ and } j = 1 \dots, J_i^n, \\ \mu_r^{\tau, \omega}, & \text{if } \omega \in A_m. \end{cases} \quad (6.55)$$

Let $\hat{\Theta}_m \triangleq (t, x, \mu^m, \beta\langle\mu^m\rangle)$. As $\mu^m = \mu$ on $[t, \tau]$ and thus $\beta\langle\mu^m\rangle = \beta\langle\mu\rangle$ on $[t, \tau]$, (2.15) shows that P_0^t -a.s.

$$\tilde{X}_s^{\hat{\Theta}_m} = X_s^{\hat{\Theta}_m} = X_s^\Theta = \tilde{X}_s^\Theta, \quad \forall s \in [t, \tau].$$

Thus similar to (6.38), for any \mathcal{F}_τ^t -measurable random variable ξ with $\underline{L}_\tau^{\hat{\Theta}_m} = \underline{L}_\tau^\Theta \leq \xi \leq \overline{L}_\tau^\Theta = \overline{L}_\tau^{\hat{\Theta}_m}$, P_0^t -a.s.,

$$(Y^{\hat{\Theta}_m}(\tau, \xi), Z^{\hat{\Theta}_m}(\tau, \xi), \underline{K}^{\hat{\Theta}_m}(\tau, \xi), \overline{K}^{\hat{\Theta}_m}(\tau, \xi)) = (Y^\Theta(\tau, \xi), Z^\Theta(\tau, \xi), \underline{K}^\Theta(\tau, \xi), \overline{K}^\Theta(\tau, \xi)). \quad (6.56)$$

Let $n \in \mathfrak{N}_m, i = 1 \dots, m, j = 1 \dots, J_i^n$ and $\omega \in \tilde{A}_j^{n,i} \triangleq A_j^{n,i} \setminus (\tilde{O}_\varepsilon \cup \hat{\mathcal{N}}_m)$, where $\hat{\mathcal{N}}_m$ is the P_0^t -null set such that $(\beta\langle\mu^m\rangle)^{\tau, \omega} \in \mathcal{V}^{\tau(\omega)}$ for all $\omega \in \tilde{\mathcal{N}}_m^c$ according to Proposition 4.6 (1). For $(r, \tilde{\omega}) \in [t_n, T] \times \Omega^{t_n}$, since $A_j^{n,i} \in \mathcal{F}_{t_n}^t$, Lemma 4.1 shows that $\omega \otimes_{t_n} \tilde{\omega} \in A_j^{n,i}$. Then one can deduce from (6.54) that

$$\begin{aligned} (\beta\langle\mu^m\rangle)_r^{t_n, \omega}(\tilde{\omega}) &= (\beta\langle\mu^m\rangle)_r(\omega \otimes_{t_n} \tilde{\omega}) = \beta(r, \omega \otimes_{t_n} \tilde{\omega}, \mu_r^m(\omega \otimes_{t_n} \tilde{\omega})) \\ &= \beta(r, \omega \otimes_{t_n} \tilde{\omega}, (\mu_i^n)_r(\tilde{\omega})) = \beta^{t_n, \omega}(r, \tilde{\omega}, (\mu_i^n)_r(\tilde{\omega})) = (\beta^{t_n, \omega}\langle\mu_i^n\rangle)_r(\tilde{\omega}). \end{aligned} \quad (6.57)$$

Clearly, $\omega_i^n \in F_\varepsilon^n \setminus \tilde{O}_\varepsilon \subset \{\tau = t_n\} \setminus \tilde{O}_\varepsilon$. As $\omega \in \tilde{A}_j^{n,i}$, we see that $\omega \in (\{\tau = t_n\} \cap \mathfrak{A}_j^{n,i}) \setminus \tilde{O}_\varepsilon \subset (\{\tau = t_n\} \cap O_{\delta(\mathfrak{k}_i^n)}^{t_n}(\omega_i^n)) \setminus \tilde{O}_\varepsilon$ and that $\tilde{X}_{t_n}^\Theta(\omega) \in \mathfrak{E}_j^{n,i} \subset O_{\lambda_n}(x_i^n)$. Applying (6.53) with $(\omega, x, \omega', x') = (\omega_i^n, x_i^n, \omega, \tilde{X}_{t_n}^\Theta(\omega))$, we see from (2.18) that

$$\tilde{Y}_{t_n}^{t_n, \tilde{X}_{t_n}^\Theta(\omega), \mu_i^n, \beta^{t_n, \omega}\langle\mu_i^n\rangle}(T, h(\tilde{X}_T^{t_n, \tilde{X}_{t_n}^\Theta(\omega), \mu_i^n, \beta^{t_n, \omega}\langle\mu_i^n\rangle))) \geq (\hat{w}_1(t_n, \tilde{X}_{t_n}^\Theta(\omega)) - c_0 \kappa_\psi \varepsilon^{\frac{1-\lambda_\Psi}{4\varpi}}) \vee \underline{l}(t_n, \tilde{X}_{t_n}^\Theta(\omega)).$$

Using similar arguments to those that lead to (6.39), one can deduce from (6.55) and (6.57) that

$$\begin{aligned} \left(\tilde{Y}_\tau^{\hat{\Theta}_m} \left(T, h \left(\tilde{X}_T^{\hat{\Theta}_m} \right) \right) \right) (\omega) &= \tilde{Y}_{\tau(\omega)}^{\tau(\omega), \tilde{X}_{\tau(\omega)}^{\Theta}(\omega), (\mu^m)^{\tau, \omega}, (\beta \langle \mu^m \rangle)^{\tau, \omega}} \left(T, h \left(\tilde{X}_T^{\tau(\omega), \tilde{X}_{\tau(\omega)}^{\Theta}(\omega), (\mu^m)^{\tau, \omega}, (\beta \langle \mu^m \rangle)^{\tau, \omega}} \right) \right) \\ &\geq \sum_{n \in \mathfrak{N}_m} \sum_{i=1}^m \sum_{j=1}^{J_i^n} \mathbf{1}_{\{\omega \in \tilde{A}_j^{n,i}\}} \tilde{Y}_{t_n}^{t_n, \tilde{X}_{t_n}^{\Theta}(\omega), \mu_i^n, \beta^{t_n, \omega} \langle \mu_i^n \rangle} \left(T, h \left(\tilde{X}_T^{t_n, \tilde{X}_{t_n}^{\Theta}(\omega), \mu_i^n, \beta^{t_n, \omega} \langle \mu_i^n \rangle} \right) \right) \\ &\quad + \mathbf{1}_{\{\omega \in A_m \cup \tilde{O}_\varepsilon\}} \underline{l}(\tau(\omega), \tilde{X}_{\tau(\omega)}^{\Theta}(\omega)) \geq \hat{\xi}_m(\omega), \quad \text{for } P_0^t\text{-a.s. } \omega \in \Omega^t, \end{aligned} \quad (6.58)$$

where $\hat{\xi}_m \triangleq \mathbf{1}_{A_m \cap \tilde{O}_\varepsilon} \left(\hat{w}_1(\tau, \tilde{X}_\tau^{\Theta}(\omega)) - c_0 \kappa_\psi \varepsilon^{\frac{1-\lambda\varpi}{4\varpi}} \right) \vee \underline{l}(\tau, \tilde{X}_\tau^{\Theta}) + \mathbf{1}_{A_m \cup \tilde{O}_\varepsilon} \underline{l}(\tau, \tilde{X}_\tau^{\Theta})$.

e) Similar to (6.41), we see from Proposition 1.2 that

$$|\tilde{Y}_t^{\Theta}(\tau, \hat{\xi}_m) - \tilde{Y}_t^{\Theta}(\tau, \hat{w}_1(\tau, \tilde{X}_\tau^{\Theta}))| \leq c_0 \|\hat{\xi}_m - \hat{w}_1(\tau, \tilde{X}_\tau^{\Theta})\|_{\mathbb{L}^q(\mathcal{F}_\tau^t)} \leq c_0 \|\mathbf{1}_{A_m \cup \tilde{O}_\varepsilon} (\bar{l}(\tau, \tilde{X}_\tau^{\Theta}) - \underline{l}(\tau, \tilde{X}_\tau^{\Theta}))\|_{\mathbb{L}^q(\mathcal{F}_\tau^t)} + c_0 \kappa_\psi \varepsilon^{\frac{1-\lambda\varpi}{4\varpi}}.$$

Then applying (2.17) with $(\zeta, \tau, \xi) = (\tau, T, h(\tilde{X}_T^{\hat{\Theta}_m}))$, applying Proposition 1.1 to (6.58), and using (6.56) for $\xi = \xi_m$ yield that

$$\begin{aligned} I(t, x, \beta) &\geq \tilde{Y}_t^{\hat{\Theta}_m} \left(T, h \left(\tilde{X}_T^{\hat{\Theta}_m} \right) \right) = \tilde{Y}_t^{\hat{\Theta}_m} \left(\tau, \tilde{Y}_\tau^{\hat{\Theta}_m} \left(T, h \left(\tilde{X}_T^{\hat{\Theta}_m} \right) \right) \right) \geq \tilde{Y}_t^{\hat{\Theta}_m}(\tau, \hat{\xi}_m) = \tilde{Y}_t^{\Theta}(\tau, \hat{\xi}_m) \\ &\geq \tilde{Y}_t^{\Theta}(\tau, \hat{w}_1(\tau, \tilde{X}_\tau^{\Theta})) - c_0 \|\mathbf{1}_{A_m \cup \tilde{O}_\varepsilon} (\bar{l}(\tau, \tilde{X}_\tau^{\Theta}) - \underline{l}(\tau, \tilde{X}_\tau^{\Theta}))\|_{\mathbb{L}^q(\mathcal{F}_\tau^t)} - c_0 \kappa_\psi \varepsilon^{\frac{1-\lambda\varpi}{4\varpi}}. \end{aligned} \quad (6.59)$$

Let $A_\varepsilon \triangleq \lim_{m \rightarrow \infty} \downarrow A_m = \bigcap_{m \in \mathbb{N}} A_m$. As $m \rightarrow \infty$ in (6.59), the Dominated Convergence Theorem and (6.43) show that

$$I(t, x, \beta) \geq \tilde{Y}_t^{\Theta}(\tau, \hat{w}_1(\tau, \tilde{X}_\tau^{\Theta})) - c_0 \|\mathbf{1}_{A_\varepsilon \cup \tilde{O}_\varepsilon} (\bar{l}(\tau, \tilde{X}_\tau^{\Theta}) - \underline{l}(\tau, \tilde{X}_\tau^{\Theta}))\|_{\mathbb{L}^q(\mathcal{F}_\tau^t)} - c_0 \kappa_\psi \varepsilon^{\frac{1-\lambda\varpi}{4\varpi}}. \quad (6.60)$$

Given $n \in \mathfrak{N}$ and $\omega \in F_\varepsilon^n \setminus \tilde{O}_\varepsilon \subset \{\tau = t_n\}$, since $(\omega, \tilde{X}_{t_n}(\omega)) \in (F_\varepsilon^n \setminus \tilde{O}_\varepsilon) \times \mathbb{R}^k$, there exists $i \in \mathbb{N}$, such that $(\omega, \tilde{X}_{t_n}(\omega))$ is in $\tilde{\mathfrak{D}}_i^n$ and thus further belongs to some $\mathfrak{A}_j^{n,i} \times \mathfrak{C}_j^{n,i}$, $j = 1, \dots, J_i^n$. To wit, $\omega \in \{\tau = t_n\} \cap \mathfrak{A}_j^{n,i} \cap \{\tilde{X}_{t_n} \in \mathfrak{C}_j^{n,i}\} = A_j^{n,i}$. It follows that

$$\left(\bigcup_{n \in \mathbb{N}} F_\varepsilon^n \right) \setminus \tilde{O}_\varepsilon = \bigcup_{n \in \mathbb{N}} (F_\varepsilon^n \setminus \tilde{O}_\varepsilon) = \bigcup_{n \in \mathfrak{N}} \left(F_\varepsilon^n \setminus \tilde{O}_\varepsilon \right) \subset \bigcup_{n \in \mathfrak{N}} \bigcup_{i \in \mathbb{N}} \bigcup_{j=1}^{J_i^n} A_j^{n,i} = \bigcup_{m \in \mathbb{N}} \bigcup_{n \in \mathfrak{N}} \bigcup_{i=1}^m \bigcup_{j=1}^{J_i^n} A_j^{n,i},$$

which together with (6.49) implies that

$$\begin{aligned} P_0^t(A_\varepsilon \cup \tilde{O}_\varepsilon) &= P_0^t \left(\left(\bigcup_{m \in \mathbb{N}} \bigcup_{n \in \mathfrak{N}_m} \bigcup_{i=1}^m \bigcup_{j=1}^{J_i^n} A_j^{n,i} \right)^c \cup \tilde{O}_\varepsilon \right) \leq P_0^t \left(\left(\bigcup_{n \in \mathbb{N}} F_\varepsilon^n \right)^c \right) + P_0^t(\tilde{O}_\varepsilon) \\ &= 1 - \sum_{n \in \mathbb{N}} P_0^t(F_\varepsilon^n) + P_0^t(\tilde{O}_\varepsilon) \leq C_\Psi \varepsilon^{\frac{1-\lambda\varpi}{2\lambda\varpi}} + 3\varepsilon. \end{aligned}$$

Thus, letting $\varepsilon \rightarrow 0$ in (6.60), we obtain

$$I(t, x, \beta) \geq \tilde{Y}_t^{t, x, \mu, \beta \langle \mu \rangle}(\tau, \hat{w}_1(\tau, \tilde{X}_\tau^{t, x, \mu, \beta \langle \mu \rangle})).$$

Eventually, taking supremum over $\mu \in \mathcal{U}^t$ on the right-hand-side and then taking infimum over $\beta \in \hat{\mathfrak{B}}^t$ on both sides yield that

$$\hat{w}_1(t, x) \geq \inf_{\beta \in \hat{\mathfrak{B}}^t} \sup_{\mu \in \mathcal{U}^t} \tilde{Y}_t^{t, x, \mu, \beta \langle \mu \rangle} \left(\tau_{\mu, \beta}, \hat{w}_1(\tau_{\mu, \beta}, \tilde{X}_{\tau_{\mu, \beta}}^{t, x, \mu, \beta \langle \mu \rangle}) \right).$$

3) For any $(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}) \in [0, T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{U} \times \mathbb{V}$, we define

$$\underline{l}(\bar{t}, \bar{x}) \triangleq -\bar{l}(\bar{t}, \bar{x}), \quad \bar{l}(\bar{t}, \bar{x}) \triangleq -\underline{l}(\bar{t}, \bar{x}), \quad \mathfrak{h}(\bar{x}) \triangleq -h(\bar{x}) \quad \text{and} \quad \mathfrak{f}(\bar{t}, \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}) \triangleq -f(\bar{t}, \bar{x}, -\bar{y}, -\bar{z}, \bar{u}, \bar{v}).$$

Given $\mu \in \mathcal{U}^t$ and $\nu \in \mathcal{V}^t$, we still let Θ stand for (t, x, μ, ν) and set $\underline{\mathcal{L}}_s^\Theta \triangleq \underline{l}(s, \tilde{X}_s^\Theta)$ and $\overline{\mathcal{L}}_s^\Theta \triangleq \bar{l}(s, \tilde{X}_s^\Theta)$, $s \in [t, T]$. For any \mathbf{F}^t -stopping time τ and any \mathcal{F}_τ^t -measurable random variable ξ with $\underline{\mathcal{L}}_\tau^\Theta \leq \xi \leq \overline{\mathcal{L}}_\tau^\Theta$, P_0^t -a.s.,

let $(\mathcal{Y}^\Theta(\tau, \xi), \mathcal{Z}^\Theta(\tau, \xi), \underline{\mathcal{X}}^\Theta(\tau, \xi), \overline{\mathcal{X}}^\Theta(\tau, \xi))$ denote the unique solution of the DRBSDE $(P_0^t, \xi, \mathfrak{f}_\tau^\Theta, \underline{\mathcal{Z}}_{\tau\wedge\cdot}^\Theta, \overline{\mathcal{Z}}_{\tau\wedge\cdot}^\Theta)$ in $\mathbb{G}_{\mathbf{F}^t}^q([t, T])$, where

$$\mathfrak{f}_\tau^\Theta(s, \omega, y, z) \triangleq \mathbf{1}_{\{s < \tau(\omega)\}} \mathfrak{f}\left(s, \tilde{X}_s^\Theta(\omega), y, z, \mu_s(\omega), \nu_s(\omega)\right), \quad \forall (s, \omega, y, z) \in [t, T] \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d.$$

Since $\underline{\mathcal{Z}}^\Theta = -\overline{L}^\Theta$ and $\overline{\mathcal{Z}}^\Theta = -\underline{L}^\Theta$, multiplying -1 in the DRBSDE $(P_0^t, \xi, \mathfrak{f}_\tau^\Theta, \underline{\mathcal{Z}}_{\tau\wedge\cdot}^\Theta, \overline{\mathcal{Z}}_{\tau\wedge\cdot}^\Theta)$ shows that $(-\mathcal{Y}^\Theta(\tau, \xi), -\mathcal{Z}^\Theta(\tau, \xi), -\overline{\mathcal{X}}^\Theta(\tau, \xi), -\underline{\mathcal{X}}^\Theta(\tau, \xi)) \in \mathbb{G}_{\mathbf{F}^t}^q([t, T])$ solves the DRBSDE $(P_0^t, -\xi, f_\tau^\Theta, \underline{L}_{\tau\wedge\cdot}^\Theta, \overline{L}_{\tau\wedge\cdot}^\Theta)$. To wit

$$(-\mathcal{Y}^\Theta(\tau, \xi), -\mathcal{Z}^\Theta(\tau, \xi), -\overline{\mathcal{X}}^\Theta(\tau, \xi), -\underline{\mathcal{X}}^\Theta(\tau, \xi)) = (Y^\Theta(\tau, -\xi), Z^\Theta(\tau, -\xi), \overline{K}^\Theta(\tau, -\xi), \underline{K}^\Theta(\tau, -\xi)). \quad (6.61)$$

Now let us consider the situation where player II acts first by choosing a \mathcal{V}^t -control to maximize $\widetilde{\mathcal{V}}_t^{t,x,\alpha\langle\nu\rangle,\nu}(T, \mathfrak{h}(\tilde{X}_T^{t,x,\alpha\langle\nu\rangle,\nu}))$, where $\alpha \in \mathcal{A}^t$ is player I's response. So the priority value and intrinsic priority value of player II are

$$\mathfrak{w}_2(t, x) \triangleq \inf_{\alpha \in \mathcal{A}^t} \sup_{\nu \in \mathcal{V}^t} \widetilde{\mathcal{V}}_t^{t,x,\alpha\langle\nu\rangle,\nu}(T, \mathfrak{h}(\tilde{X}_T^{t,x,\alpha\langle\nu\rangle,\nu})) \quad \text{and} \quad \overline{\mathfrak{w}}_2(t, x) \triangleq \inf_{\alpha \in \hat{\mathcal{A}}^t} \sup_{\nu \in \mathcal{V}^t} \widetilde{\mathcal{V}}_t^{t,x,\alpha\langle\nu\rangle,\nu}(T, \mathfrak{h}(\tilde{X}_T^{t,x,\alpha\langle\nu\rangle,\nu})).$$

For any family $\{\tau_{\nu,\alpha} : \nu \in \mathcal{V}^t, \alpha \in \mathcal{A}^t\}$ of $\mathbb{Q}_{t,T}$ -valued, \mathbf{F}^t -stopping times, applying (2.30) yields that

$$\mathfrak{w}_2(t, x) \leq \inf_{\alpha \in \mathcal{A}^t} \sup_{\nu \in \mathcal{V}^t} \widetilde{\mathcal{V}}_t^{t,x,\alpha\langle\nu\rangle,\nu}(\tau_{\nu,\alpha}, \mathfrak{w}_2(\tau_{\nu,\alpha}, \tilde{X}_{\tau_{\nu,\alpha}}^{t,x,\alpha\langle\nu\rangle,\nu})). \quad (6.62)$$

For any $(t, x) \in [0, T] \times \mathbb{R}^k$, we see from (6.61) that

$$-\mathfrak{w}_2(t, x) = \sup_{\alpha \in \mathcal{A}^t} \inf_{\nu \in \mathcal{V}^t} -\widetilde{\mathcal{V}}_t^{t,x,\alpha\langle\nu\rangle,\nu}(T, \mathfrak{h}(\tilde{X}_T^{t,x,\alpha\langle\nu\rangle,\nu})) = \sup_{\alpha \in \mathcal{A}^t} \inf_{\nu \in \mathcal{V}^t} \widetilde{Y}_t^{t,x,\alpha\langle\nu\rangle,\nu}(T, \mathfrak{h}(\tilde{X}_T^{t,x,\alpha\langle\nu\rangle,\nu})) = w_2(t, x).$$

Putting it back into (6.62) and using (6.61), we obtain (2.32). Similarly, we have (2.33) and its inverse holds under (\mathbf{U}_λ) for some $\lambda \in (0, 1)$. \square

6.3 Proofs of Section 3

The proof of Theorem 3.1 relies on the following comparison theorem for generalized reflected BSDEs.

Proposition 6.1. *Given $t \in [0, T]$ and $i = 1, 2$, let $\mathfrak{f}_i : [t, T] \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a $\mathcal{P}(\overline{\mathbf{F}}^t) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)/\mathcal{B}(\mathbb{R})$ -measurable function. For some $\xi_i \in \mathbb{L}^q(\overline{\mathcal{F}}_T^t)$ and $L^i \in \mathbb{C}_{\mathbf{F}^t}^{+,q}([t, T])$ (resp. $\mathbb{C}_{\mathbf{F}^t}^{-,q}([t, T])$) with $\xi_i \geq$ (resp. \leq) L_T^i , P_0^t -a.s., let $(Y^i, Z^i, V^i, K^i) \in \mathbb{C}_{\mathbf{F}^t}^q([t, T]) \times \mathbb{H}_{\mathbf{F}^t}^{2,q}([t, T], \mathbb{R}^d) \times \mathcal{V}_{\mathbf{F}^t}([t, T]) \times \mathbb{K}_{\mathbf{F}^t}([t, T])$ be a solution of the following generalized reflected backward stochastic differential equation with lower (resp. upper) obstacle on the probability space $(\Omega^t, \overline{\mathcal{F}}_T^t, P_0^t)$ ($\underline{\text{RBSDE}}(P_0^t, \xi_i, \mathfrak{f}_i, L^i)$, resp. $\overline{\text{RBSDE}}(P_0^t, \xi_i, \mathfrak{f}_i, L^i)$, for short):*

$$\begin{cases} L_s^i \leq (\text{resp. } \geq) Y_s^i = \xi_i + \int_s^T \mathfrak{f}_i(r, Y_r^i, Z_r^i) dr + V_T^i - V_s^i + (\text{resp. } -) (K_T^i - K_s^i) - \int_s^T Z_r^i dB_r^t, & s \in [t, T], \\ \int_t^T (Y_s^i - L_s^i) dK_s^i = 0. \end{cases} \quad (6.63)$$

If $P_0^t(\xi_1 \leq \xi_2) = P_0^t(L_1^1 \leq L_1^2, \forall s \in [t, T]) = 1$, if $V^1 - V^2$ is a decreasing process, and if either of the following two conditions holds:

(i) \mathfrak{f}_1 satisfies (1.6) and $\mathfrak{f}_1(s, Y_s^2, Z_s^2) \leq \mathfrak{f}_2(s, Y_s^2, Z_s^2)$, $ds \times dP_0^t$ -a.s.,

(ii) \mathfrak{f}_2 satisfies (1.6) and $\mathfrak{f}_1(s, Y_s^1, Z_s^1) \leq \mathfrak{f}_2(s, Y_s^1, Z_s^1)$, $ds \times dP_0^t$ -a.s.;

then $P_0^t(Y_s^1 \leq Y_s^2, \forall s \in [t, T]) = 1$.

Proof: We first show the comparison for $\underline{\text{RBSDE}}(P_0^t, \xi_i, f_i, L^i)$, $i = 1, 2$: Let $\Delta \mathfrak{X} \triangleq \mathfrak{X}^1 - \mathfrak{X}^2$ for $\mathfrak{X} = Y, Z, V$. Similar to (6.6), applying Tanaka's formula to process $(\Delta Y)^+$ and using Corollary 1 of [24], we obtain

$$\begin{aligned} & |(\Delta Y_s)^+|^q - |(\Delta Y_{s'})^+|^q + \frac{q(q-1)}{2} \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_s)^+|^{q-2} |\Delta Z_r|^2 dr \\ & \leq q \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{q-1} (f_1(r, Y_r^1, Z_r^1) - f_2(r, Y_r^2, Z_r^2)) dr + q \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{q-1} (d\Delta V_r + dK_r^1 - dK_r^2) \\ & \quad - q \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{q-1} \Delta Z_r dB_r^t - \frac{q}{2} \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{q-1} d\mathfrak{L}_r, \quad \forall t \leq s \leq s' \leq T, \end{aligned} \quad (6.64)$$

where \mathfrak{L} is a real-valued, \mathbf{F}^t -adapted, increasing and continuous process known as “local time”. The flat-off condition of (Y^1, Z^1, V^1, K^1) implies that P_0^t -a.s.

$$0 \leq \int_t^T \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{q-1} dK_r^1 = \int_t^T \mathbf{1}_{\{L_r^1 = Y_r^1 > Y_r^2\}} |(L_r^1 - Y_r^2)^+|^{q-1} dK_r^1 \leq \int_t^T \mathbf{1}_{\{L_r^1 > L_r^2\}} |(L_r^1 - L_r^2)^+|^{q-1} dK_r^1 = 0.$$

Putting this back into (6.64) and using Lipschitz continuity of f_1 in (y, z) yield that

$$\begin{aligned} & |(\Delta Y_s)^+|^q - |(\Delta Y_{s'})^+|^q + \frac{q(q-1)}{2} \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_s)^+|^{q-2} |\Delta Z_r|^2 dr \\ & \leq q \int_s^{s'} \mathbf{1}_{\{\Delta Y_r > 0\}} |(\Delta Y_r)^+|^{q-1} \left[(\gamma |\Delta Y_r| + \gamma |\Delta Z_r| + (f_1(r, Y_r^2, Z_r^2) - f_2(r, Y_r^2, Z_r^2))^+) dr - \Delta Z_r dB_r^t \right], \quad \forall t \leq s \leq s' \leq T, \end{aligned}$$

which is similar to (6.7) except that ϖ is specified by q . Then using similar arguments to those that lead to (6.13), we can deduce that

$$0 \leq E_t \left[\sup_{s \in [t, T]} |(\Delta Y_s)^+|^q \right] \leq c_0 E_t \left[|(\xi_1 - \xi_2)^+|^q \right] + c_0 E_t \left[\left(\int_t^T (f_1(r, Y_r^1, Z_r^1) - f_2(r, Y_r^2, Z_r^2))^+ dr \right)^q \right] = 0.$$

Therefore, it holds P_0^t -a.s. that $(\Delta Y_s)^+ = 0$, or $Y_s^1 \leq Y_s^2$ for any $s \in [t, T]$.

Next, we consider the case of reflected BSDEs with upper obstacles: For either $i = 1$ or $i = 2$, as (Y^i, Z^i, V^i, K^i) solves $\overline{\text{RBSDE}}(P_0^t, \xi_i, f_i, L^i)$, the quadruplet $(\hat{Y}^i, \hat{Z}^i, \hat{V}^i, \hat{K}^i) \triangleq (-Y^{3-i}, -Z^{3-i}, -V^{3-i}, -K^{3-i})$ solves the $\underline{\text{RBSDE}}(P_0^t, \hat{\xi}_i, \hat{f}_i, \hat{L}^i)$ with $\hat{\xi}_i \triangleq -\xi_{3-i}$, $\hat{L}^i \triangleq -L_{s-i}^{3-i}$ and the generator

$$\hat{f}_i(s, \omega, y, z) \triangleq -f_{3-i}(s, \omega, -y, -z), \quad \forall (s, \omega, y, z) \in [t, T] \times \Omega^t \times \mathbb{R} \times \mathbb{R}^d.$$

It holds P_0^t -a.s. that $\hat{\xi}_1 - \hat{\xi}_2 = -\xi_2 + \xi_1 \leq 0$ and $\hat{L}_s^1 - \hat{L}_s^2 = -L_s^2 + L_s^1 \leq 0$ for any $s \in [t, T]$. Also, the process $\hat{V}^1 - \hat{V}^2 = -V^2 + V^1$ is decreasing. For either $i = 1$ or $i = 2$, if f_i satisfies (1.6) and $f_1(s, Y_s^{3-i}, Z_s^{3-i}) \leq f_2(s, Y_s^{3-i}, Z_s^{3-i})$, $ds \times dP_0^t$ -a.s., then \hat{f}_{3-i} satisfies (1.6) and $\hat{f}_1(s, \hat{Y}_s^i, \hat{Z}_s^i) - \hat{f}_2(s, \hat{Y}_s^i, \hat{Z}_s^i) = -f_2(s, Y_s^{3-i}, Z_s^{3-i}) + f_1(s, Y_s^{3-i}, Z_s^{3-i}) \leq 0$, $ds \times dP_0^t$ -a.s. Hence, all conditions to compare $\underline{\text{RBSDE}}(P_0^t, \hat{\xi}_1, \hat{f}_1, \hat{L}^1)$ with $\underline{\text{RBSDE}}(P_0^t, \hat{\xi}_2, \hat{f}_2, \hat{L}^2)$ are satisfied. Then we can conclude that P_0^t -a.s., $\hat{Y}_s^1 - \hat{Y}_s^2 = -Y_s^2 + Y_s^1$ for any $s \in [t, T]$. \square

Proof of Theorem 3.1:

1) We first show that \bar{w}_1 is a viscosity subsolution of (3.1) with Hamiltonian \bar{H}_1 when $\mathbb{U}_0 = \bigcup_{i \in \mathbb{N}} F_i$ for closed subsets $\{F_i\}_{i \in \mathbb{N}}$ of \mathbb{U} . Let $(t_0, x_0, \varphi) \in (0, T) \times \mathbb{R}^k \times \mathbb{C}^{1,2}([0, T] \times \mathbb{R}^k)$ be such that $\bar{w}_1(t_0, x_0) = \varphi(t_0, x_0)$ and that $\bar{w}_1 - \varphi$ attains a strict local maximum at (t_0, x_0) , i.e., for some $\delta_0 \in (0, t_0 \wedge (T - t_0))$

$$(\bar{w}_1 - \varphi)(t, x) < (\bar{w}_1 - \varphi)(t_0, x_0) = 0, \quad \forall (t, x) \in O_{\delta_0}(t_0, x_0) \setminus \{(t_0, x_0)\}. \quad (6.65)$$

Let us simply denote $(\varphi(t_0, x_0), D_x \varphi(t_0, x_0), D_x^2 \varphi(t_0, x_0))$ by (y_0, z_0, Γ_0) . Since $\underline{l}(t_0, x_0) \leq \varphi(t_0, x_0) = \bar{w}_1(t_0, x_0) \leq \bar{l}(t_0, x_0)$ by (2.27), it is clear that

$$\min \left\{ (\varphi - \underline{l})(t_0, x_0), \max \left\{ -\frac{\partial \varphi}{\partial t}(t_0, x_0) - \bar{H}_1(t_0, x_0, y_0, z_0, \Gamma_0), (\varphi - \bar{l})(t_0, x_0) \right\} \right\} \leq 0$$

if $\overline{H}_1(t_0, x_0, y_0, z_0, \Gamma_0) = \infty$.

To draw a contradiction, we assume that when $\overline{H}_1(t_0, x_0, y_0, z_0, \Gamma_0) < \infty$,

$$\varrho \triangleq \min \left\{ (\varphi - \underline{l})(t_0, x_0), \max \left\{ -\frac{\partial \varphi}{\partial t}(t_0, x_0) - \overline{H}_1(t_0, x_0, y_0, z_0, \Gamma_0), (\varphi - \overline{l})(t_0, x_0) \right\} \right\} > 0. \quad (6.66)$$

Then the continuity of φ and \overline{l} implies that for some $\delta_1 \in (0, \delta_0)$

$$(\varphi - \underline{l})(t, x) \geq \frac{3}{4} \varrho, \quad \forall (t, x) \in \overline{O}_{\delta_1}(t_0, x_0). \quad (6.67)$$

As $\varphi(t_0, x_0) \leq \overline{l}(t_0, x_0)$, we also see from (6.66) that $-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \overline{H}_1(t_0, x_0, y_0, z_0, \Gamma_0) \geq \varrho$. Thus, one can find an $m \in \mathbb{N}$ such that

$$-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{7}{8} \varrho \geq \sup_{u \in \mathbb{U}_0} \inf_{v \in \mathcal{O}_u^m} \overline{\lim}_{\mathbb{U}_0 \ni u' \rightarrow u} \sup_{(t, x, y, z, \Gamma) \in O_{1/m}(t_0, x_0, y_0, z_0, \Gamma_0)} H(t, x, y, z, \Gamma, u', v). \quad (6.68)$$

As $\varphi \in \mathbb{C}^{1,2}([0, T] \times \mathbb{R}^k)$, there exists a $\delta < \frac{1}{2m} \wedge \delta_1$ such that for any $(t, x) \in \overline{O}_\delta(t_0, x_0)$

$$\left| \frac{\partial \varphi}{\partial t}(t, x) - \frac{\partial \varphi}{\partial t}(t_0, x_0) \right| \leq \frac{1}{8} \varrho \quad (6.69)$$

$$\text{and} \quad \left| \varphi(t, x) - \varphi(t_0, x_0) \right| \vee \left| D_x \varphi(t, x) - D_x \varphi(t_0, x_0) \right| \vee \left| D_x^2 \varphi(t, x) - D_x^2 \varphi(t_0, x_0) \right| \leq \frac{1}{2m},$$

the latter of which together with (6.68) implies that

$$-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{7}{8} \varrho \geq \sup_{u \in \mathbb{U}_0} \inf_{v \in \mathcal{O}_u^m} \overline{\lim}_{\mathbb{U}_0 \ni u' \rightarrow u} \sup_{(t, x) \in \overline{O}_\delta(t_0, x_0)} H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), u', v).$$

Then for any $u \in \mathbb{U}_0$, there exists a $\mathfrak{P}(u) \in \mathcal{O}_u^m$ such that

$$-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{3}{4} \varrho \geq \overline{\lim}_{\mathbb{U}_0 \ni u' \rightarrow u} \sup_{(t, x) \in \overline{O}_\delta(t_0, x_0)} H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), u', \mathfrak{P}(u))$$

and we can find a $\lambda(u) \in (0, 1)$ such that for any $u' \in \mathbb{U}_0 \cap O_{\lambda(u)}(u)$

$$-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{5}{8} \varrho \geq \sup_{(t, x) \in \overline{O}_\delta(t_0, x_0)} H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), u', \mathfrak{P}(u)). \quad (6.70)$$

Let $\mathfrak{D}(u) \triangleq O_{\lambda(u)}(u)$, $u \in \mathbb{U}_0$. For any $i \in \mathbb{N}$, $\{\mathfrak{D}(u)\}_{u \in F_i}$ together with $\mathbb{U} \setminus F_i$ form an open cover of \mathbb{U} . Since the separable metric space \mathbb{U} is Lindelöf, one can find a sequence $\{u_j^i\}_{j \in \mathbb{N}}$ of F_i such that $F_i \subset \bigcup_{j \in \mathbb{N}} \mathfrak{D}(u_j^i)$. Let $\{u_\ell\}_{\ell \in \mathbb{N}}$ represent the countable set $\{u_j^i\}_{i, j \in \mathbb{N}} \subset \mathbb{U}_0$ and let \hat{v} be an arbitrary element of \mathbb{V}_0 . It is clear that

$$\mathfrak{P}(u) \triangleq \sum_{\ell \in \mathbb{N}} \mathbf{1}_{\{u \in \mathfrak{D}(u_\ell) \setminus \bigcup_{\ell' < \ell} \mathfrak{D}(u_{\ell'})\}} \mathfrak{P}(u_\ell) + \mathbf{1}_{\{u \in \mathbb{U} \setminus \bigcup_{\ell \in \mathbb{N}} \mathfrak{D}(u_\ell)\}} \hat{v} \in \mathbb{V}_0, \quad \forall u \in \mathbb{U}$$

defines a $\mathcal{B}(\mathbb{U})/\mathcal{B}(\mathbb{V})$ -measurable function.

For any $u \in \mathbb{U}_0$, since $\mathbb{U}_0 = \bigcup_{i \in \mathbb{N}} F_i \subset \bigcup_{i, j \in \mathbb{N}} \mathfrak{D}(u_j^i) = \bigcup_{\ell \in \mathbb{N}} \mathfrak{D}(u_\ell)$, there exists a $\ell \in \mathbb{N}$ such that $u \in \mathfrak{D}(u_\ell) \setminus \bigcup_{\ell' < \ell} \mathfrak{D}(u_{\ell'})$.

We see from (6.70) that

$$\begin{aligned} -\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{5}{8} \varrho &\geq \sup_{(t, x) \in \overline{O}_\delta(t_0, x_0)} H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), u, \mathfrak{P}(u_\ell)) \\ &= \sup_{(t, x) \in \overline{O}_\delta(t_0, x_0)} H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), u, \mathfrak{P}(u)), \end{aligned}$$

which together with (6.69) implies that

$$-\frac{\partial \varphi}{\partial t}(t, x) - \frac{1}{2} \varrho \geq H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), u, \mathfrak{P}(u)), \quad \forall (t, x) \in \overline{O}_\delta(t_0, x_0), \quad \forall u \in \mathbb{U}_0. \quad (6.71)$$

Let $\wp \triangleq \inf \{(\varphi - \bar{w}_1)(t, x) : (t, x) \in \bar{O}_\delta(t_0, x_0) \setminus O_{\frac{\delta}{2}}(t_0, x_0)\}$. Since the set $\bar{O}_\delta(t_0, x_0) \setminus O_{\frac{\delta}{2}}(t_0, x_0)$ is compact, there exists a sequence $\{(t_n, x_n)\}_{n \in \mathbb{N}}$ of $\bar{O}_\delta(t_0, x_0) \setminus O_{\frac{\delta}{2}}(t_0, x_0)$ that converges to some $(t_*, x_*) \in \bar{O}_\delta(t_0, x_0) \setminus O_{\frac{\delta}{2}}(t_0, x_0)$ and satisfies $\wp = \lim_{n \rightarrow \infty} \downarrow (\varphi - \bar{w}_1)(t_n, x_n)$. The continuity of φ and the upper semicontinuity of \bar{w}_1 imply that $\varphi - \bar{w}_1$ is also lower semicontinuous. Thus, it follows that $\wp \leq (\varphi - \bar{w}_1)(t_*, x_*) \leq \lim_{n \rightarrow \infty} \downarrow (\varphi - \bar{w}_1)(t_n, x_n) = \wp$, which together with (6.65) shows that

$$\wp = \min \{(\varphi - \bar{w}_1)(t, x) : (t, x) \in \bar{O}_\delta(t_0, x_0) \setminus O_{\frac{\delta}{2}}(t_0, x_0)\} = (\varphi - \bar{w}_1)(t_*, x_*) > 0. \quad (6.72)$$

Then we set $\tilde{\wp} \triangleq \frac{\wp \wedge \varrho}{2(1 \vee \gamma)T} > 0$ and let $\{(t_j, x_j)\}_{j \in \mathbb{N}}$ be a sequence of $O_{\frac{\delta}{4}}(t_0, x_0)$ such that

$$\lim_{j \rightarrow \infty} (t_j, x_j) = (t_0, x_0) \quad \text{and} \quad \lim_{j \rightarrow \infty} \hat{w}_1(t_j, x_j) = \bar{w}_1(t_0, x_0) = \varphi(t_0, x_0).$$

As $\lim_{j \rightarrow \infty} (\hat{w}_1(t_j, x_j) - \varphi(t_j, x_j)) = 0$, it holds for some $j \in \mathbb{N}$ that

$$|\hat{w}_1(t_j, x_j) - \varphi(t_j, x_j)| < \frac{1}{2} \tilde{\wp} t_0. \quad (6.73)$$

In particular, $\mathfrak{P}(t, \omega, u) \triangleq \mathfrak{P}(u)$, $(t, \omega, u) \in [t_j, T] \times \Omega^{t_j} \times \mathbb{U}$ is a \mathbb{V}_0 -valued, $\mathcal{P}(\mathbf{F}^{t_j}) \otimes \mathcal{B}(\mathbb{U})/\mathcal{B}(\mathbb{V})$ -measurable function. For any $(t, \omega, u) \in [t_j, T] \times \Omega^{t_j} \times \mathbb{U}$, if $u \in \mathfrak{D}_i \setminus \bigcup_{j < i} \mathfrak{D}_j$ for some $i \in \mathbb{N}$, then

$$[\mathfrak{P}(t, \omega, u)]_{\mathbb{V}} = [\mathfrak{P}(u)]_{\mathbb{V}} \leq m + m[u_i]_{\mathbb{U}} \leq m + m[u]_{\mathbb{U}} + m\rho_{\mathbb{U}}(u, u_i) < m + m[u]_{\mathbb{U}} + m\lambda(u_i) < 2m + m[u]_{\mathbb{U}};$$

otherwise, $[\mathfrak{P}(t, \omega, u)]_{\mathbb{V}} = [\hat{v}]_{\mathbb{V}}$. This shows that \mathfrak{P} satisfies (2.25) with $\Psi = 2m + [\hat{v}]_{\mathbb{V}}$ and $\kappa = m$. Clearly, (2.26) automatically holds for \mathfrak{P} . Hence, $\mathfrak{P} \in \hat{\mathfrak{B}}^{t_j}$.

For any $\mu \in \mathcal{U}^{t_j}$, we set $\Theta_\mu \triangleq (t_j, x_j, \mu, \mathfrak{P}(\mu))$ and define two \mathbf{F}^{t_j} -stopping times:

$$\tau_\mu \triangleq \inf \left\{ s \in (t_j, T] : (s, \tilde{X}_s^{\Theta_\mu}) \notin \bar{O}_{\frac{3}{4}\delta}(t_0, x_0) \right\} \quad \text{and} \quad \zeta_\mu \triangleq \inf \left\{ s \in (\tau_\mu, T] : (s, \tilde{X}_s^{\Theta_\mu}) \notin \bar{O}_\delta(t_0, x_0) \setminus O_{\frac{\delta}{2}}(t_0, x_0) \right\} \wedge T.$$

Since $|(T, \tilde{X}_T^{\Theta_\mu}) - (t_0, x_0)| \geq T - t_0 > \delta_0 > \delta_1 > \frac{3}{4}\delta$, one can deduce from the continuity of \tilde{X}^{Θ_μ} that

$$\tau_\mu < T \quad \text{and} \quad (\tau_\mu, \tilde{X}_{\tau_\mu}^{\Theta_\mu}) \in \partial O_{\frac{3}{4}\delta}(t_0, x_0), \quad P_0^{t_j} - a.s. \quad (6.74)$$

Given $n \in \mathbb{N}$, we define $q^n(s) \triangleq \frac{\lfloor 2^n s \rfloor}{2^n} \wedge T$, $s \in [0, T]$. Then $\tau_\mu^n \triangleq q^n(\tau_\mu) \wedge \zeta_\mu$ is an \mathbf{F}^{t_j} -stopping time. Applying (2.17) with $(\zeta, \tau, \xi) = (\tau_\mu^n, q^n(\tau_\mu), \hat{w}_1(q^n(\tau_\mu), \tilde{X}_{q^n(\tau_\mu)}^{\Theta_\mu}))$, we can deduce from Proposition 1.2 and Hölder's inequality that

$$\begin{aligned} & \left| \tilde{Y}_{t_j}^{\Theta_\mu}(\tau_\mu^n, \hat{w}_1(\tau_\mu^n, \tilde{X}_{\tau_\mu^n}^{\Theta_\mu})) - \tilde{Y}_{t_j}^{\Theta_\mu}(q^n(\tau_\mu), \hat{w}_1(q^n(\tau_\mu), \tilde{X}_{q^n(\tau_\mu)}^{\Theta_\mu})) \right|^{\frac{q+1}{2}} \\ &= \left| \tilde{Y}_{t_j}^{\Theta_\mu}(\tau_\mu^n, \hat{w}_1(\tau_\mu^n, \tilde{X}_{\tau_\mu^n}^{\Theta_\mu})) - \tilde{Y}_{t_j}^{\Theta_\mu}(\tau_\mu^n, \tilde{Y}_{\tau_\mu^n}^{\Theta_\mu}(q^n(\tau_\mu), \hat{w}_1(q^n(\tau_\mu), \tilde{X}_{q^n(\tau_\mu)}^{\Theta_\mu}))) \right|^{\frac{q+1}{2}} \\ &\leq c_0 E_{t_j} \left[\left| \hat{w}_1(\tau_\mu^n, \tilde{X}_{\tau_\mu^n}^{\Theta_\mu}) - \tilde{Y}_{\tau_\mu^n}^{\Theta_\mu}(q^n(\tau_\mu), \hat{w}_1(q^n(\tau_\mu), \tilde{X}_{q^n(\tau_\mu)}^{\Theta_\mu})) \right|^{\frac{q+1}{2}} \right] \\ &= c_0 E_{t_j} \left[\mathbf{1}_{\{q^n(\tau_\mu) > \zeta_\mu\}} \left| \hat{w}_1(\zeta_\mu, \tilde{X}_{\zeta_\mu}^{\Theta_\mu}) - \tilde{Y}_{\tau_\mu^n}^{\Theta_\mu}(q^n(\tau_\mu), \hat{w}_1(q^n(\tau_\mu), \tilde{X}_{q^n(\tau_\mu)}^{\Theta_\mu})) \right|^{\frac{q+1}{2}} \right] \\ &\leq c_0 \left(P_0^t(q^n(\tau_\mu) > \zeta_\mu) \right)^{\frac{q-1}{2q}} \left\{ E_{t_j} \left[\left| \hat{w}_1(\zeta_\mu, \tilde{X}_{\zeta_\mu}^{\Theta_\mu}) - \tilde{Y}_{\tau_\mu^n}^{\Theta_\mu}(q^n(\tau_\mu), \hat{w}_1(q^n(\tau_\mu), \tilde{X}_{q^n(\tau_\mu)}^{\Theta_\mu})) \right|^q \right] \right\}^{\frac{q+1}{2q}}. \quad (6.75) \end{aligned}$$

By (2.3),

$$\left| \hat{w}_1(\zeta_\mu, \tilde{X}_{\zeta_\mu}^{\Theta_\mu}) \right| \leq (|\underline{l}| \vee |\bar{l}|) \left(\zeta_\mu, \tilde{X}_{\zeta_\mu}^{\Theta_\mu} \right) \leq \underline{L}_* + \bar{l}_* + \gamma \sup_{s \in [t_j, T]} |\tilde{X}_s^{\Theta_\mu}|^{2/q}, \quad (6.76)$$

where $\underline{L}_* \triangleq \sup_{s \in [t, T]} |\underline{L}(s, 0)|$ and $\bar{L}_* \triangleq \sup_{s \in [t, T]} |\bar{L}(s, 0)|$. Similarly,

$$\left| \tilde{Y}_{\tau_\mu^n}^{\Theta_\mu} \left(q^n(\tau_\mu), \hat{w}_1(q^n(\tau_\mu), \tilde{X}_{q^n(\tau_\mu)}^{\Theta_\mu}) \right) \right| \leq \left(|\underline{L}_{\tau_\mu^n}^{\Theta_\mu}| \vee |\bar{L}_{\tau_\mu^n}^{\Theta_\mu}| \right) = (|\underline{L}| \vee |\bar{L}|) \left(\tau_\mu^n, \tilde{X}_{\tau_\mu^n}^{\Theta_\mu} \right) \leq \underline{L}_* + \bar{L}_* + \gamma \sup_{s \in [t_j, T]} |\tilde{X}_s^{\Theta_\mu}|^{2/q}.$$

Putting it and (6.76) back into (6.75) gives that

$$\begin{aligned} & \left| \tilde{Y}_{t_j}^{\Theta_\mu} \left(\tau_\mu^n, \hat{w}_1(\tau_\mu^n, \tilde{X}_{\tau_\mu^n}^{\Theta_\mu}) \right) - \tilde{Y}_{t_j}^{\Theta_\mu} \left(q^n(\tau_\mu), \hat{w}_1(q^n(\tau_\mu), \tilde{X}_{q^n(\tau_\mu)}^{\Theta_\mu}) \right) \right|^{\frac{q+1}{2}} \\ & \leq c_0 \left(P_0^t(q^n(\tau_\mu) > \zeta_\mu) \right)^{\frac{q-1}{2q}} \left\{ 1 + E_{t_j} \left[\sup_{s \in [t_j, T]} |\tilde{X}_s^{\Theta_\mu}|^2 \right] \right\}^{\frac{q+1}{2q}}. \end{aligned} \quad (6.77)$$

Since $\tau_\mu < \zeta_\mu$, P_0^t -a.s. by (6.74) and since $\lim_{n \rightarrow \infty} \downarrow q^n(\tau_\mu) = \tau_\mu$, we see from (2.7) that the right-hand-side of (6.77) converges to 0 as $n \rightarrow \infty$. Hence, for some $n_\mu \in \mathbb{N}$

$$\left| \tilde{Y}_{t_j}^{\Theta_\mu} \left(\hat{\tau}_\mu, \hat{w}_1(\hat{\tau}_\mu, \tilde{X}_{\hat{\tau}_\mu}^{\Theta_\mu}) \right) - \tilde{Y}_{t_j}^{\Theta_\mu} \left(q^{n_\mu}(\tau_\mu), \hat{w}_1(q^{n_\mu}(\tau_\mu), \tilde{X}_{q^{n_\mu}(\tau_\mu)}^{\Theta_\mu}) \right) \right| < \frac{1}{4} \tilde{\wp} t_0, \quad (6.78)$$

where $\hat{\tau}_\mu \triangleq \tau_\mu^{n_\mu} = q^{n_\mu}(\tau_\mu) \wedge \zeta_\mu$.

As $\hat{\tau}_\mu$ is an \mathbf{F}^{t_j} -stopping time, the continuity of φ and \tilde{X}^{Θ_μ} show that $\mathcal{Y}_s^\mu \triangleq \varphi(\hat{\tau}_\mu \wedge s, \tilde{X}_{\hat{\tau}_\mu \wedge s}^{\Theta_\mu}) - \tilde{\wp}(\hat{\tau}_\mu \wedge s)$, $s \in [t_j, T]$ defines a real-valued, \mathbf{F}^{t_j} -adapted continuous process. Applying Itô's formula to \mathcal{Y}^μ yields that

$$\mathcal{Y}_s^\mu = \mathcal{Y}_T^\mu + \int_s^T \mathfrak{f}_r^\mu dr - \int_s^T \mathcal{Z}_r^\mu dB_r^{t_j}, \quad s \in [t_j, T], \quad (6.79)$$

where $\mathcal{Z}_r^\mu \triangleq \mathbf{1}_{\{r < \hat{\tau}_\mu\}} D_x \varphi(r, \tilde{X}_r^{\Theta_\mu}) \cdot \sigma(r, \tilde{X}_r^{\Theta_\mu}, \mu_r, (\mathfrak{P}(\mu))_r)$ and

$$\mathfrak{f}_r^\mu \triangleq \mathbf{1}_{\{r < \hat{\tau}_\mu\}} \left\{ \tilde{\wp} - \frac{\partial \varphi}{\partial t}(r, \tilde{X}_r^{\Theta_\mu}) - D_x \varphi(r, \tilde{X}_r^{\Theta_\mu}) \cdot b(r, \tilde{X}_r^{\Theta_\mu}, \mu_r, (\mathfrak{P}(\mu))_r) - \frac{1}{2} \text{trace} \left(\sigma \sigma^T(r, \tilde{X}_r^{\Theta_\mu}, \mu_r, (\mathfrak{P}(\mu))_r) \cdot D_x^2 \varphi(r, \tilde{X}_r^{\Theta_\mu}) \right) \right\}.$$

As $\varphi \in \mathbb{C}^{1,2}([t, T] \times \mathbb{R}^k)$, the measurability of b , σ and \mathfrak{P} show that both \mathcal{Z}^μ and \mathfrak{f}^μ are \mathbf{F}^{t_j} -progressively measurable.

Since it holds P_0^t -a.s. that

$$(\hat{\tau}_\mu \wedge s, \tilde{X}_{\hat{\tau}_\mu \wedge s}^{\Theta_\mu}) \in \overline{\mathcal{O}}_\delta(t_0, x_0), \quad \forall s \in [t_j, T], \quad (6.80)$$

we see from the continuity of φ that \mathcal{Y}^μ is a bounded process, and we can deduce from (2.1), (2.2) as well as Hölder's inequality that

$$\begin{aligned} E_{t_j} \left[\left(\int_{t_j}^T |\mathcal{Z}_s^\mu|^2 ds \right)^{q/2} \right] &= (\gamma C_\varphi)^q E_{t_j} \left[\left(\int_{t_j}^{\hat{\tau}_\mu} \left(1 + |\tilde{X}_s^{\Theta_\mu}| + [\mu_s]_{\mathbb{U}} + [(\mathfrak{P}(\hat{\mu}))_s]_{\mathbb{V}} \right)^2 ds \right)^{q/2} \right] \\ &\leq c_0 C_\varphi^q \left\{ (1 + |x_0| + \delta)^q + \left(E_{t_j} \int_{t_j}^T [\mu_s]_{\mathbb{U}}^2 ds \right)^{q/2} + \left(E_{t_j} \int_{t_j}^T [(\mathfrak{P}(\hat{\mu}))_s]_{\mathbb{V}}^2 ds \right)^{q/2} \right\} < \infty, \text{ i.e. } \mathcal{Z}^\mu \in \mathbb{H}_{\mathbf{F}^{t_j}}^{2,q}([t, T], \mathbb{R}^d), \end{aligned} \quad (6.81)$$

where $C_\varphi \triangleq \sup_{(t,x) \in \overline{\mathcal{O}}_\delta(t_0, x_0)} |D_x \varphi(t, x)|$. Moreover, (6.80) and (6.67) imply that P_0^t -a.s.

$$\mathcal{Y}_s^\mu \geq \underline{L}(\hat{\tau}_\mu \wedge s, \tilde{X}_{\hat{\tau}_\mu \wedge s}^{\Theta_\mu}) + \frac{3}{4} \varrho - \tilde{\wp} T > \underline{L}(\hat{\tau}_\mu \wedge s, \tilde{X}_{\hat{\tau}_\mu \wedge s}^{\Theta_\mu}) = \underline{L}_{\hat{\tau}_\mu \wedge s}^{\Theta_\mu}, \quad \forall s \in [t_j, T],$$

which together with (6.79) shows that $\{(\mathcal{Y}_s^\mu, \mathcal{Z}_s^\mu, 0, 0)\}_{s \in [t_j, T]}$ solves the $\underline{\text{RBSDE}}(P_0^{t_j}, \mathcal{Y}_T^\mu, \mathfrak{f}^\mu, \underline{L}_{\hat{\tau}_\mu \wedge \cdot}^{\Theta_\mu})$ (see (6.63)).

Since $\tau_\mu \leq \hat{\tau}_\mu \leq \zeta_\mu$ and since $\bar{w}_1(t, x) \geq \hat{w}_1(t, x)$ for any $(t, x) \in [0, T] \times \mathbb{R}^k$, we can deduce from (6.72) that

$$\mathcal{Y}_T^\mu \geq \varphi(\hat{\tau}_\mu, \tilde{X}_{\hat{\tau}_\mu}^{\Theta_\mu}) - \tilde{\wp} T > \varphi(\hat{\tau}_\mu, \tilde{X}_{\hat{\tau}_\mu}^{\Theta_\mu}) - \wp \geq \bar{w}_1(\hat{\tau}_\mu, \tilde{X}_{\hat{\tau}_\mu}^{\Theta_\mu}) \geq \hat{w}_1(\hat{\tau}_\mu, \tilde{X}_{\hat{\tau}_\mu}^{\Theta_\mu}).$$

Also, (6.80), (6.71) and (2.5) show that for $ds \times dP_0^{t_j}$ -a.s. $(s, \omega) \in [t_j, T] \times \Omega^{t_j}$

$$\begin{aligned} f_s^\mu(\omega) &\geq \mathbf{1}_{\{s < \hat{\tau}_\mu(\omega)\}} \left\{ \tilde{\varphi} + \frac{1}{2} \varrho + f\left(s, \omega, \tilde{X}_s^{\Theta_\mu}(\omega), \mathcal{Y}_s^\mu(\omega) - \tilde{\varphi}s, \mathcal{Z}_s^\mu(\omega), \mu_s(\omega), (\mathfrak{P}\langle\mu\rangle)_s(\omega)\right) \right\} \\ &\geq \mathbf{1}_{\{s < \hat{\tau}_\mu(\omega)\}} \left\{ \tilde{\varphi} + \frac{1}{2} \varrho - \gamma \tilde{\varphi} T + f\left(s, \omega, \tilde{X}_s^{\Theta_\mu}(\omega), \mathcal{Y}_s^\mu(\omega), \mathcal{Z}_s^\mu(\omega), \mu_s(\omega), (\mathfrak{P}\langle\mu\rangle)_s(\omega)\right) \right\} \\ &\geq f_{\hat{\tau}_\mu}^{\Theta_\mu}(s, \omega, \mathcal{Y}_s^\mu(\omega), \mathcal{Z}_s^\mu(\omega)). \end{aligned}$$

Clearly, $\left(\tilde{Y}^{\Theta_\mu}(\hat{\tau}_\mu, \hat{w}_1(\hat{\tau}_\mu, \tilde{X}_{\hat{\tau}_\mu}^{\Theta_\mu})), \tilde{Z}^{\Theta_\mu}(\hat{\tau}_\mu, \hat{w}_1(\hat{\tau}_\mu, \tilde{X}_{\hat{\tau}_\mu}^{\Theta_\mu})), -\tilde{K}^{\Theta_\mu}(\hat{\tau}_\mu, \hat{w}_1(\hat{\tau}_\mu, \tilde{X}_{\hat{\tau}_\mu}^{\Theta_\mu}), \tilde{K}^{\Theta_\mu}(\hat{\tau}_\mu, \hat{w}_1(\hat{\tau}_\mu, \tilde{X}_{\hat{\tau}_\mu}^{\Theta_\mu})))\right)$ solves $\underline{\text{RBSDE}}\left(P_0^{t_j}, \hat{w}_1(\hat{\tau}_\mu, \tilde{X}_{\hat{\tau}_\mu}^{\Theta_\mu}), f_{\hat{\tau}_\mu}^{\Theta_\mu}, \underline{L}_{\hat{\tau}_\mu}^{\Theta_\mu}\right)$. As $f_{\hat{\tau}_\mu}^{\Theta_\mu}$ is Lipschitz continuous in (y, z) , we know from Proposition 6.1 that $P_0^{t_j}$ -a.s.

$$\mathcal{Y}_s^\mu \geq \tilde{Y}_s^{\Theta_\mu}(\hat{\tau}_\mu, \hat{w}_1(\hat{\tau}_\mu, \tilde{X}_{\hat{\tau}_\mu}^{\Theta_\mu})), \quad \forall s \in [t_j, T].$$

Letting $s = t_j$ and using (6.78), we obtain

$$\begin{aligned} \varphi(t_j, x_j) - \frac{3}{4} \tilde{\varphi} t_0 &> \varphi(t_j, x_j) - \tilde{\varphi} t_j = \mathcal{Y}_{t_j}^\mu \geq \tilde{Y}_{t_j}^{t_j, x_j, \mu, \mathfrak{P}\langle\mu\rangle}(\hat{\tau}_\mu, \hat{w}_1(\hat{\tau}_\mu, \tilde{X}_{\hat{\tau}_\mu}^{t_j, x_j, \mu, \mathfrak{P}\langle\mu\rangle})) \\ &> \tilde{Y}_{t_j}^{t_j, x_j, \mu, \mathfrak{P}\langle\mu\rangle}(q^{n_\mu}(\tau_\mu), \hat{w}_1(q^{n_\mu}(\tau_\mu), \tilde{X}_{q^{n_\mu}(\tau_\mu)}^{t_j, x_j, \mu, \mathfrak{P}\langle\mu\rangle})) - \frac{1}{4} \tilde{\varphi} t_0, \end{aligned}$$

where we used the fact that $t_j > t_0 - \frac{1}{4}\delta > t_0 - \frac{1}{4}\delta_0 > \frac{3}{4}t_0$. Taking supremum over $\mu \in \mathcal{U}^{t_j}$ gives that

$$\varphi(t_j, x_j) - \frac{3}{4} \tilde{\varphi} t_0 \geq \sup_{\mu \in \mathcal{U}^{t_j}} \tilde{Y}_{t_j}^{t_j, x_j, \mu, \mathfrak{P}\langle\mu\rangle}(q^{n_\mu}(\tau_\mu), \hat{w}_1(q^{n_\mu}(\tau_\mu), \tilde{X}_{q^{n_\mu}(\tau_\mu)}^{t_j, x_j, \mu, \mathfrak{P}\langle\mu\rangle})) - \frac{1}{4} \tilde{\varphi} t_0. \quad (6.82)$$

Let $\{\tau_{\mu, \beta} : \mu \in \mathcal{U}^{t_j}, \beta \in \hat{\mathfrak{B}}^{t_j}\}$ be an arbitrary family of $\mathbb{Q}_{t_j, T}$ -valued, \mathbf{F}^{t_j} -stopping times such that $\tau_{\mu, \mathfrak{P}} = q^{n_\mu}(\tau_\mu)$ for any $\mu \in \mathcal{U}^{t_j}$. Then (6.82), (6.73) and Theorem 2.1 imply that

$$\begin{aligned} \varphi(t_j, x_j) - \frac{3}{4} \tilde{\varphi} t_0 &\geq \sup_{\mu \in \mathcal{U}^{t_j}} \tilde{Y}_{t_j}^{t_j, x_j, \mu, \mathfrak{P}\langle\mu\rangle}(q^{n_\mu}(\tau_\mu), \hat{w}_1(q^{n_\mu}(\tau_\mu), \tilde{X}_{q^{n_\mu}(\tau_\mu)}^{t_j, x_j, \mu, \mathfrak{P}\langle\mu\rangle})) - \frac{1}{4} \tilde{\varphi} t_0 \\ &\geq \inf_{\beta \in \hat{\mathfrak{B}}^{t_j}} \sup_{\mu \in \mathcal{U}^{t_j}} \tilde{Y}_{t_j}^{t_j, x_j, \mu, \mathfrak{P}\langle\mu\rangle}(q^{n_\mu}(\tau_\mu), \hat{w}_1(q^{n_\mu}(\tau_\mu), \tilde{X}_{q^{n_\mu}(\tau_\mu)}^{t_j, x_j, \mu, \mathfrak{P}\langle\mu\rangle})) - \frac{1}{4} \tilde{\varphi} t_0 \\ &\geq \hat{w}_1(t_j, x_j) - \frac{1}{4} \tilde{\varphi} t_0 > \varphi(t_j, x_j) - \frac{3}{4} \tilde{\varphi} t_0. \end{aligned}$$

A contradiction appears. Therefore, \bar{w}_1 is a viscosity subsolution of (3.1) with Hamiltonian \bar{H}_1 .

Similarly, one can show that w_1^* is also a viscosity subsolution of (3.1) with Hamiltonian \bar{H}_1 when $\mathbb{U}_0 = \bigcup_{i \in \mathbb{N}} F_i$ for closed subsets $\{F_i\}_{i \in \mathbb{N}}$ of \mathbb{U} . Using the transformation similar to that in the part (3) of proof of Theorem 2.1, one can show that if \mathbb{V}_0 is a countable union of closed subsets of \mathbb{V} , then \underline{w}_2 and w_2^* are two viscosity supersolutions of (3.1) with Hamiltonian \underline{H}_2 .

2) Next, by assuming (\mathbf{V}_λ) for some $\lambda \in (0, 1)$, we shall show that \underline{w}_1 is a viscosity supersolution of (3.1) with Hamiltonian \underline{H}_1 . Let $(t_0, x_0, \varphi) \in (0, T) \times \mathbb{R}^k \times \mathbb{C}^{1,2}([0, T] \times \mathbb{R}^k)$ be such that $\underline{w}_1(t_0, x_0) = \varphi(t_0, x_0)$ and that $\underline{w}_1 - \varphi$ attains a strict local minimum at (t_0, x_0) , i.e., for some $\delta_0 \in (0, t_0 \wedge (T - t_0))$

$$(\underline{w}_1 - \varphi)(t, x) > (\underline{w}_1 - \varphi)(t_0, x_0) = 0, \quad \forall (t, x) \in O_{\delta_0}(t_0, x_0) \setminus \{(t_0, x_0)\}.$$

The continuity of \underline{l}, \bar{l} and (2.27) imply that $\underline{l}(t, x) \leq \underline{w}_1(t, x) \leq \bar{l}(t, x)$, $\forall (t, x) \in [0, T] \times \mathbb{R}^k$. So it suffices to show that

$$\max\left\{-\frac{\partial}{\partial t}\varphi(t_0, x_0) - \underline{H}_1(t_0, x_0, \varphi(t_0, x_0), D_x\varphi(t_0, x_0), D_x^2\varphi(t_0, x_0)), (\varphi - \bar{l})(t_0, x_0)\right\} \geq 0, \quad (6.83)$$

which clearly holds if $\underline{H}_1(t_0, x_0, \varphi(t_0, x_0), D_x\varphi(t_0, x_0), D_x^2\varphi(t_0, x_0)) = -\infty$.

To make a contradiction, we assume that when $\underline{H}_1(t_0, x_0, \varphi(t_0, x_0), D_x \varphi(t_0, x_0), D_x^2 \varphi(t_0, x_0)) > -\infty$, (6.83) does not hold, i.e.

$$\varrho \triangleq \min \left\{ \frac{\partial}{\partial t} \varphi(t_0, x_0) + \underline{H}_1(t_0, x_0, \varphi(t_0, x_0), D_x \varphi(t_0, x_0), D_x^2 \varphi(t_0, x_0)), (\bar{l} - \varphi)(t_0, x_0) \right\} > 0.$$

As $\varphi \in \mathbb{C}^{1,2}([0, T] \times \mathbb{R}^k)$, there exists a $\hat{u} \in \mathbb{U}_0$ such that

$$\liminf_{(t,x) \rightarrow (t_0, x_0)} \inf_{v \in \mathbb{V}_0} H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), \hat{u}, v) \geq \frac{3}{4} \varrho - \frac{\partial}{\partial t} \varphi(t_0, x_0).$$

Then by the continuity of φ and \bar{l} , one can find a $\delta \in (0, \delta_0)$ such that

$$\inf_{v \in \mathbb{V}_0} H(t, x, \varphi(t, x), D_x \varphi(t, x), D_x^2 \varphi(t, x), \hat{u}, v) \geq \frac{1}{2} \varrho - \frac{\partial}{\partial t} \varphi(t, x) \quad \text{and} \quad (\bar{l} - \varphi)(t, x) \geq \frac{3}{4} \varrho, \quad \forall (t, x) \in \overline{O}_\delta(t_0, x_0). \quad (6.84)$$

Similar to (6.72) we set $\wp \triangleq \min \{(\underline{w}_1 - \varphi)(t, x) : (t, x) \in \overline{O}_\delta(t_0, x_0) \setminus O_{\frac{\delta}{2}}(t_0, x_0)\}$ and $\tilde{\wp} \triangleq \frac{\wp \wedge \varrho}{2(1 \vee \gamma)T} > 0$. Let $\{(t_j, x_j)\}_{j \in \mathbb{N}}$ be a sequence of $O_{\frac{\delta}{4}}(t_0, x_0)$ such that

$$\lim_{j \rightarrow \infty} (t_j, x_j) = (t_0, x_0) \quad \text{and} \quad \lim_{j \rightarrow \infty} \hat{w}_1(t_j, x_j) = \underline{w}_1(t_0, x_0) = \varphi(t_0, x_0).$$

As $\lim_{j \rightarrow \infty} (\hat{w}_1(t_j, x_j) - \varphi(t_j, x_j)) = 0$, it holds for some $j \in \mathbb{N}$ that

$$|\hat{w}_1(t_j, x_j) - \varphi(t_j, x_j)| < \frac{1}{2} \tilde{\wp} t_0. \quad (6.85)$$

Clearly, $\hat{\mu}_s \triangleq \hat{u}$, $s \in [t_j, T]$ is a constant \mathcal{U}^{t_j} -control. Fix $\beta \in \hat{\mathfrak{B}}^{t_j}$. We set $\Theta_\beta \triangleq (t_j, x_j, \hat{\mu}, \beta \langle \hat{\mu} \rangle)$ and define two \mathbf{F}^{t_j} -stopping times:

$$\tau_\beta \triangleq \inf \left\{ s \in (t_j, T] : (s, \tilde{X}_s^{\Theta_\beta}) \notin \overline{O}_{\frac{\delta}{4}}(t_0, x_0) \right\} \quad \text{and} \quad \zeta_\beta \triangleq \inf \left\{ s \in (\tau_\beta, T] : (s, \tilde{X}_s^{\Theta_\beta}) \notin \overline{O}_\delta(t_0, x_0) \setminus O_{\frac{\delta}{2}}(t_0, x_0) \right\} \wedge T.$$

Since $|(T, \tilde{X}_T^{\Theta_\beta}) - (t_0, x_0)| \geq T - t_0 > \delta_0 > \frac{3}{4} \delta$, one can deduce from the continuity of \tilde{X}^{Θ_β} that

$$\tau_\beta < T \quad \text{and} \quad (\tau_\beta, \tilde{X}_{\tau_\beta}^{\Theta_\beta}) \in \partial O_{\frac{\delta}{4}}(t_0, x_0), \quad P_0^{t_j} - a.s. \quad (6.86)$$

Given $n \in \mathbb{N}$, we define $q^n(s) \triangleq \lfloor \frac{2^n s}{2^n} \rfloor \wedge T$, $s \in [0, T]$. Then $\tau_\beta^n \triangleq q^n(\tau_\beta) \wedge \zeta_\beta$ is an \mathbf{F}^{t_j} -stopping time. Similar to (6.77), applying (2.17) with $(\zeta, \tau, \xi) = (\tau_\beta^n, q^n(\tau_\beta), \hat{w}_1(q^n(\tau_\beta), \tilde{X}_{q^n(\tau_\beta)}^{\Theta_\beta}))$, we can deduce from Proposition 1.2, Hölder's inequality and (2.3) that

$$\begin{aligned} & \left| \tilde{Y}_{t_j}^{\Theta_\beta} \left(\tau_\beta^n, \hat{w}_1(\tau_\beta^n, \tilde{X}_{\tau_\beta^n}^{\Theta_\beta}) \right) - \tilde{Y}_{t_j}^{\Theta_\beta} \left(q^n(\tau_\beta), \hat{w}_1(q^n(\tau_\beta), \tilde{X}_{q^n(\tau_\beta)}^{\Theta_\beta}) \right) \right|^{\frac{q+1}{2}} \\ & \leq c_0 \left(P_0^t(q^n(\tau_\beta) > \zeta_\beta) \right)^{\frac{q-1}{2q}} \left\{ 1 + E_{t_j} \left[\sup_{s \in [t_j, T]} |\tilde{X}_s^{\Theta_\beta}|^2 \right] \right\}^{\frac{q+1}{2q}}. \end{aligned} \quad (6.87)$$

Since $\tau_\beta < \zeta_\beta$, P_0^t -a.s. by (6.86) and since $\lim_{n \rightarrow \infty} q^n(\tau_\beta) = \tau_\beta$, we see from (2.7) that the right-hand-side of (6.87) converges to 0 as $n \rightarrow \infty$. Hence, for some $n_\beta \in \mathbb{N}$

$$\left| \tilde{Y}_{t_j}^{\Theta_\beta} \left(\hat{\tau}_\beta, \hat{w}_1(\hat{\tau}_\beta, \tilde{X}_{\hat{\tau}_\beta}^{\Theta_\beta}) \right) - \tilde{Y}_{t_j}^{\Theta_\beta} \left(q^{n_\beta}(\tau_\beta), \hat{w}_1(q^{n_\beta}(\tau_\beta), \tilde{X}_{q^{n_\beta}(\tau_\beta)}^{\Theta_\beta}) \right) \right| < \frac{1}{4} \tilde{\wp} t_0, \quad (6.88)$$

where $\hat{\tau}_\beta \triangleq \tau_\beta^{n_\beta} = q^{n_\beta}(\tau_\beta) \wedge \zeta_\beta$.

As $\hat{\tau}_\beta$ is an \mathbf{F}^{t_j} -stopping time, the continuity of φ and \tilde{X}^{Θ_β} show that $\mathcal{Y}_s^\beta \triangleq \varphi(\hat{\tau}_\beta \wedge s, \tilde{X}_{\hat{\tau}_\beta \wedge s}^{\Theta_\beta}) + \tilde{\wp}(\hat{\tau}_\beta \wedge s)$, $s \in [t_j, T]$ defines a real-valued, \mathbf{F}^{t_j} -adapted continuous process. Applying Itô's formula to \mathcal{Y}^β yields that

$$\mathcal{Y}_s^\beta = \mathcal{Y}_T^\beta + \int_s^T \mathfrak{f}_r^\beta dr - \int_s^T \mathcal{Z}_r^\beta dB_r^{t_j}, \quad s \in [t_j, T], \quad (6.89)$$

where $\mathcal{Z}_r^\beta \triangleq \mathbf{1}_{\{r < \hat{\tau}_\beta\}} D_x \varphi(r, \tilde{X}_r^{\Theta_\beta}) \cdot \sigma(r, \tilde{X}_r^{\Theta_\beta}, \hat{u}, (\beta \langle \hat{\mu} \rangle)_r)$ and

$$\mathfrak{f}_r^\beta \triangleq -\mathbf{1}_{\{r < \hat{\tau}_\beta\}} \left\{ \tilde{\varphi} + \frac{\partial \varphi}{\partial t}(r, \tilde{X}_r^{\Theta_\beta}) + D_x \varphi(r, \tilde{X}_r^{\Theta_\beta}) \cdot b(r, \tilde{X}_r^{\Theta_\beta}, \hat{u}, (\beta \langle \hat{\mu} \rangle)_r) + \frac{1}{2} \text{trace} \left(\sigma \sigma^T(r, \tilde{X}_r^{\Theta_\beta}, \hat{u}, (\beta \langle \hat{\mu} \rangle)_r) \cdot D_x^2 \varphi(r, \tilde{X}_r^{\Theta_\beta}) \right) \right\}.$$

As $\varphi \in \mathbb{C}^{1,2}([t, T] \times \mathbb{R}^k)$, the measurability of b, σ and β show that both \mathcal{Z}^β and \mathfrak{f}^β are \mathbf{F}^{t_j} -progressively measurable.

Since it holds P_0^t -a.s. that

$$(\hat{\tau}_\beta \wedge s, \tilde{X}_{\hat{\tau}_\beta \wedge s}^{\Theta_\beta}) \in \overline{O}_\delta(t_0, x_0), \quad \forall s \in [t_j, T], \quad (6.90)$$

we see from the continuity of φ that \mathcal{Y}^β is a bounded process. And similar to (6.81), we can deduce from (2.1), (2.2) as well as Hölder's inequality that $\mathcal{Z}^\beta \in \mathbb{H}_{\mathbf{F}^{t_j}}^{2,q}([t, T], \mathbb{R}^d)$. Moreover, (6.90) and (6.84) imply that P_0^t -a.s.

$$\mathcal{Y}_s^\beta \leq \bar{l}(\hat{\tau}_\beta \wedge s, \tilde{X}_{\hat{\tau}_\beta \wedge s}^{\Theta_\beta}) - \frac{3}{4} \varrho + \tilde{\varphi} T < \bar{l}(\hat{\tau}_\beta \wedge s, \tilde{X}_{\hat{\tau}_\beta \wedge s}^{\Theta_\beta}) = \bar{L}_{\hat{\tau}_\beta \wedge s}^{\Theta_\beta}, \quad \forall s \in [t_j, T],$$

which together with (6.89) shows that $\{(\mathcal{Y}_s^\beta, \mathcal{Z}_s^\beta, 0, 0)\}_{s \in [t_j, T]}$ solves the $\overline{\text{RBSDE}}(P_0^{t_j}, \mathcal{Y}_T^\beta, \mathfrak{f}_T^\beta, \bar{L}_{\hat{\tau}_\beta \wedge \cdot}^{\Theta_\beta})$.

Since $\tau_\beta \leq \hat{\tau}_\beta \leq \zeta_\beta$ and since $\underline{w}_1(t, x) \leq \hat{w}_1(t, x)$ for any $(t, x) \in [0, T] \times \mathbb{R}^k$, we can deduce that

$$\mathcal{Y}_T^\beta \leq \varphi(\hat{\tau}_\beta, \tilde{X}_{\hat{\tau}_\beta}^{\Theta_\beta}) + \tilde{\varphi} T < \varphi(\hat{\tau}_\beta, \tilde{X}_{\hat{\tau}_\beta}^{\Theta_\beta}) + \varphi \leq \underline{w}_1(\hat{\tau}_\beta, \tilde{X}_{\hat{\tau}_\beta}^{\Theta_\beta}) \leq \hat{w}_1(\hat{\tau}_\beta, \tilde{X}_{\hat{\tau}_\beta}^{\Theta_\beta}).$$

Also, (6.90), (6.84) and (2.5) show that for $ds \times dP_0^{t_j}$ -a.s. $(s, \omega) \in [t_j, T] \times \Omega^{t_j}$

$$\begin{aligned} \mathfrak{f}_s^\beta(\omega) &\leq \mathbf{1}_{\{s < \hat{\tau}_\beta(\omega)\}} \left\{ -\tilde{\varphi} - \frac{1}{2} \varrho + f(s, \omega, \tilde{X}_s^{\Theta_\beta}(\omega), \mathcal{Y}_s^\beta(\omega) - \tilde{\varphi} s, \mathcal{Z}_s^\beta(\omega), \hat{u}, (\beta \langle \hat{\mu} \rangle)_s(\omega)) \right\} \\ &\leq \mathbf{1}_{\{s < \hat{\tau}_\beta(\omega)\}} \left\{ -\tilde{\varphi} - \frac{1}{2} \varrho + \gamma \tilde{\varphi} T + f(s, \omega, \tilde{X}_s^{\Theta_\beta}(\omega), \mathcal{Y}_s^\beta(\omega), \mathcal{Z}_s^\beta(\omega), \hat{u}, (\beta \langle \hat{\mu} \rangle)_s(\omega)) \right\} \\ &\leq f_{\hat{\tau}_\beta}^{\Theta_\beta}(s, \omega, \mathcal{Y}_s^\beta(\omega), \mathcal{Z}_s^\beta(\omega)). \end{aligned}$$

Clearly, $(\tilde{Y}^{\Theta_\beta}(\hat{\tau}_\beta, \hat{w}_1(\hat{\tau}_\beta, \tilde{X}_{\hat{\tau}_\beta}^{\Theta_\beta})), \tilde{Z}^{\Theta_\beta}(\hat{\tau}_\beta, \hat{w}_1(\hat{\tau}_\beta, \tilde{X}_{\hat{\tau}_\beta}^{\Theta_\beta})), \tilde{K}^{\Theta_\beta}(\hat{\tau}_\beta, \hat{w}_1(\hat{\tau}_\beta, \tilde{X}_{\hat{\tau}_\beta}^{\Theta_\beta})), \tilde{K}^{\Theta_\beta}(\hat{\tau}_\beta, \hat{w}_1(\hat{\tau}_\beta, \tilde{X}_{\hat{\tau}_\beta}^{\Theta_\beta})))$ solves $\overline{\text{RBSDE}}(P_0^{t_j}, \hat{w}_1(\hat{\tau}_\beta, \tilde{X}_{\hat{\tau}_\beta}^{\Theta_\beta}), f_{\hat{\tau}_\beta}^{\Theta_\beta}, \bar{L}_{\hat{\tau}_\beta \wedge \cdot}^{\Theta_\beta})$. As $f_{\hat{\tau}_\beta}^{\Theta_\beta}$ is Lipschitz continuous in (y, z) , we know from Proposition 6.1 that $P_0^{t_j}$ -a.s.

$$\mathcal{Y}_s^\beta \leq \tilde{Y}_s^{\Theta_\beta}(\hat{\tau}_\beta, \hat{w}_1(\hat{\tau}_\beta, \tilde{X}_{\hat{\tau}_\beta}^{\Theta_\beta})), \quad \forall s \in [t_j, T].$$

Letting $s = t_j$ and using (6.88), we obtain

$$\begin{aligned} \varphi(t_j, x_j) + \frac{3}{4} \tilde{\varphi} t_0 &< \varphi(t_j, x_j) + \tilde{\varphi} t_j = \mathcal{Y}_{t_j}^\beta \leq \tilde{Y}_{t_j}^{t_j, x_j, \hat{\mu}, \beta \langle \hat{\mu} \rangle}(\hat{\tau}_\beta, \hat{w}_1(\hat{\tau}_\beta, \tilde{X}_{\hat{\tau}_\beta}^{t_j, x_j, \hat{\mu}, \beta \langle \hat{\mu} \rangle})) \\ &< \tilde{Y}_{t_j}^{t_j, x_j, \hat{\mu}, \beta \langle \hat{\mu} \rangle}(q^{n_\beta}(\tau_\beta), \hat{w}_1(q^{n_\beta}(\tau_\beta), \tilde{X}_{q^{n_\beta}(\tau_\beta)}^{t_j, x_j, \hat{\mu}, \beta \langle \hat{\mu} \rangle})) + \frac{1}{4} \tilde{\varphi} t_0, \end{aligned} \quad (6.91)$$

where we used the fact that $t_j > t_0 - \frac{1}{4} \delta > t_0 - \frac{1}{4} \delta_0 > \frac{3}{4} t_0$.

Let $\{\tau_{\mu, \beta} : \mu \in \mathcal{U}^{t_j}, \beta \in \hat{\mathfrak{B}}^{t_j}\}$ be an arbitrary family of $\mathbb{Q}_{t_j, T}$ -valued, \mathbf{F}^{t_j} -stopping times such that $\tau_{\hat{\mu}, \beta} = q^{n_\beta}(\tau_\beta)$ for any $\beta \in \hat{\mathfrak{B}}^{t_j}$. By (6.91), it holds for any $\beta \in \hat{\mathfrak{B}}^{t_j}$ that

$$\begin{aligned} \varphi(t_j, x_j) + \frac{3}{4} \tilde{\varphi} t_0 &< \tilde{Y}_{t_j}^{t_j, x_j, \hat{\mu}, \beta \langle \hat{\mu} \rangle}(\tau_{\hat{\mu}, \beta}, \hat{w}_1(\tau_{\hat{\mu}, \beta}, \tilde{X}_{\tau_{\hat{\mu}, \beta}}^{t_j, x_j, \hat{\mu}, \beta \langle \hat{\mu} \rangle})) + \frac{1}{4} \tilde{\varphi} t_0 \\ &\leq \sup_{\mu \in \mathcal{U}^{t_j}} \tilde{Y}_{t_j}^{t_j, x_j, \mu, \beta \langle \mu \rangle}(\tau_{\mu, \beta}, \hat{w}_1(\tau_{\mu, \beta}, \tilde{X}_{\tau_{\mu, \beta}}^{t_j, x_j, \mu, \beta \langle \mu \rangle})) + \frac{1}{4} \tilde{\varphi} t_0. \end{aligned}$$

Then taking infimum over $\beta \in \hat{\mathfrak{B}}^{t_j}$, we see from (6.85) and Theorem 2.1 that

$$\varphi(t_j, x_j) + \frac{3}{4} \tilde{\varphi} t_0 \leq \inf_{\beta \in \hat{\mathfrak{B}}^{t_j}} \sup_{\mu \in \mathcal{U}^{t_j}} \tilde{Y}_{t_j}^{t_j, x_j, \mu, \beta \langle \mu \rangle}(\tau_{\mu, \beta}, \hat{w}_1(\tau_{\mu, \beta}, \tilde{X}_{\tau_{\mu, \beta}}^{t_j, x_j, \mu, \beta \langle \mu \rangle})) + \frac{1}{4} \tilde{\varphi} t_0 \leq \hat{w}_1(t_j, x_j) + \frac{1}{4} \tilde{\varphi} t_0 < \varphi(t_j, x_j) + \frac{3}{4} \tilde{\varphi} t_0.$$

A contradiction appears. Therefore, \underline{w}_1 is a viscosity supersolution of (3.1) with Hamiltonian \underline{H}_1 under (\mathbf{V}_λ) .

Using the transformation similar to that in the part (3) of proof of Theorem 2.1, one can show that \overline{w}_2 is a viscosity subsolution of (3.1) with Hamiltonian \overline{H}_2 given (\mathbf{U}_λ) for some $\lambda \in (0, 1)$. \square

6.4 Proofs of Section 4

Proof of Lemma 4.1: Set $\Lambda \triangleq \left\{ A \subset \Omega^t : A = \bigcup_{\omega \in A} (\omega \otimes_s \Omega^s) \right\}$. For any $A \in \Lambda$, we claim that

$$\omega \otimes_s \Omega^s \subset A^c \text{ for any } \omega \in A^c. \quad (6.92)$$

Assume not, there is an $\omega \in A^c$ and an $\tilde{\omega} \in \Omega^s$ such that $\omega \otimes_s \tilde{\omega} \in A$, thus $(\omega \otimes_s \tilde{\omega}) \otimes_s \Omega^s \subset A$. Then $\omega \in \omega \otimes_s \Omega^s = (\omega \otimes_s \tilde{\omega}) \otimes_s \Omega^s \subset A$. A contradiction appear.

For any $r \in [t, s]$ and $\mathcal{E} \in \mathcal{B}(\mathbb{R}^d)$, if $\omega \in (B_r^t)^{-1}(\mathcal{E})$, then for any $\tilde{\omega} \in \Omega^s$, $(\omega \otimes_s \tilde{\omega})(r) = \omega(r) \in \mathcal{E}$, i.e., $\omega \otimes_s \tilde{\omega} \in (B_r^t)^{-1}(\mathcal{E})$. Thus $\omega \otimes_s \Omega^s \subset (B_r^t)^{-1}(\mathcal{E})$, which implies that $(B_r^t)^{-1}(\mathcal{E}) \in \Lambda$. In particular, $\emptyset \in \Lambda$ and $\Omega^t \in \Lambda$. For any $A \in \Lambda$, (6.92) implies that $A^c \in \Lambda$. For any $\{A_n\}_{n \in \mathbb{N}} \subset \Lambda$, $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} \left(\bigcup_{\omega \in A_n} (\omega \otimes_s \Omega^s) \right) = \bigcup_{\omega \in \bigcup_{n \in \mathbb{N}} A_n} (\omega \otimes_s \Omega^s)$, namely, $\bigcup_{n \in \mathbb{N}} A_n \in \Lambda$. Thus, Λ is a σ -field of Ω^t containing all generating sets of \mathcal{F}_s^t . It then follows that $\mathcal{F}_s^t \subset \Lambda$, proving the lemma. \square

Proof of Lemma 4.2: Let A be an open subset of Ω^t . Given $\tilde{\omega} \in A^{s, \omega}$, there exists a δ such that $O_\delta(\omega \otimes_s \tilde{\omega}) \subset A$. For any $\tilde{\omega}' \in O_\delta(\tilde{\omega})$, one can deduce that $\sup_{r \in [t, T]} |(\omega \otimes_s \tilde{\omega}')(r) - (\omega \otimes_s \tilde{\omega})(r)| = \sup_{r \in [s, T]} |\tilde{\omega}'(r) - \tilde{\omega}(r)| < \delta$, which shows that $\omega \otimes_s \tilde{\omega}' \in O_\delta(\omega \otimes_s \tilde{\omega}) \subset A$, i.e. $\tilde{\omega}' \in A^{s, \omega}$. Hence, $A^{s, \omega}$ is an open subset of Ω^s . If A is a closed subset of Ω^t , then $(A^c)^{s, \omega}$ is an open subset of Ω^s and it follows from (4.2) that $A^{s, \omega} = ((A^c)^{s, \omega})^c$ is a closed subset of Ω^s .

Next, let $r \in [s, T]$. For any $t' \in [t, r]$ and $\mathcal{E} \in \mathcal{B}(\mathbb{R}^d)$, we can deduce that

$$\left((B_{t'}^t)^{-1}(\mathcal{E}) \right)^{s, \omega} = \begin{cases} \Omega^s, & \text{if } t' \in [t, s) \text{ and } \omega(t') \in \mathcal{E}; \\ \emptyset, & \text{if } t' \in [t, s) \text{ and } \omega(t') \notin \mathcal{E}; \\ \{\tilde{\omega} \in \Omega^s : \omega(s) + \tilde{\omega}(t_i) \in \mathcal{E}\} = (B_{t_i}^s)^{-1}(\mathcal{E}') \in \mathcal{F}_r^s, & \text{if } t' \in [s, r], \end{cases}$$

where $\mathcal{E}' = \{x - \omega(s) : x \in \mathcal{E}\} \in \mathcal{B}(\mathbb{R}^d)$. Thus all the generating sets of \mathcal{F}_r^t belong to $\Lambda \triangleq \left\{ A \subset \Omega^t : A^{s, \omega} \in \mathcal{F}_r^s \right\}$. In particular, $\emptyset, \Omega^t \in \Lambda$. For any $A \in \Lambda$ and $\{A_n\}_{n \in \mathbb{N}} \subset \Lambda$, we see from (4.2) and (4.4) that $(A^c)^{s, \omega} = (A^{s, \omega})^c \in \mathcal{F}_r^s$ and $\left(\bigcup_{n \in \mathbb{N}} A_n \right)^{s, \omega} = \bigcup_{n \in \mathbb{N}} A_n^{s, \omega} \in \mathcal{F}_r^s$, i.e. $A^c, \bigcup_{n \in \mathbb{N}} A_n \in \Lambda$. So Λ is a σ -field of Ω^t , it follows that $\mathcal{F}_r^t \subset \Lambda$, i.e., $A^{s, \omega} \in \mathcal{F}_r^s$ for any $A \in \mathcal{F}_r^t$.

On the other hand, since the continuity of paths in Ω^t shows that

$$\omega \otimes_s \Omega^s = \left\{ \omega' \in \Omega^t : \omega'(t') = \omega(t'), \forall t' \in \mathbb{Q} \cap [t, s] \right\} = \bigcap_{t' \in \mathbb{Q} \cap [t, s]} (B_{t'}^t)^{-1}(\omega(t')) \in \mathcal{F}_s^t. \quad (6.93)$$

For any $\tilde{A} \in \mathcal{F}_r^s$, applying Lemma 1.2 with $S = T$ gives that $\Pi_{t,s}^{-1}(\tilde{A}) \in \mathcal{F}_r^t$, which together with (6.93) shows that $\omega \otimes_s \tilde{A} = \Pi_{t,s}^{-1}(\tilde{A}) \cap (\omega \otimes_s \Omega^s) \in \mathcal{F}_r^t$. \square

Proof of Proposition 4.1: Let ξ be a \mathcal{F}_r^t -measurable random variable for some $r \in [s, T]$. For any $\mathcal{M} \in \mathcal{B}(\mathbb{M})$, since $\xi^{-1}(\mathcal{M}) \in \mathcal{F}_r^t$, Lemma 4.2 shows that

$$(\xi^{s, \omega})^{-1}(\mathcal{M}) = \{\tilde{\omega} \in \Omega^s : \xi(\omega \otimes_s \tilde{\omega}) \in \mathcal{M}\} = \{\tilde{\omega} \in \Omega^s : \omega \otimes_s \tilde{\omega} \in \xi^{-1}(\mathcal{M})\} = (\xi^{-1}(\mathcal{M}))^{s, \omega} \in \mathcal{F}_r^s. \quad (6.94)$$

Thus $\xi^{s, \omega}$ is \mathcal{F}_r^s -measurable. Next, consider a \mathbb{M} -valued, \mathbf{F}^t -adapted process $\{X_r\}_{r \in [t, T]}$. For any $r \in [s, T]$ and $\mathcal{M} \in \mathcal{B}(\mathbb{M})$, similar to (6.94), one can deduce that $(X_r^{s, \omega})^{-1}(\mathcal{M}) = (X_r^{-1}(\mathcal{M}))^{s, \omega} \in \mathcal{F}_r^s$, which shows that $\{X_r^{s, \omega}\}_{r \in [s, T]}$ is \mathbf{F}^s -adapted. \square

Proof of Proposition 4.2: For any $\mathcal{E} \in \mathcal{B}([t, T_0])$ and $A \in \mathcal{F}_{T_0}^t$, Lemma 4.2 shows that

$$(\mathcal{E} \times A)^{s, \omega} = \{(r, \tilde{\omega}) \in [s, T_0] \times \Omega^s : (r, \omega \otimes_s \tilde{\omega}) \in \mathcal{E} \times A\} = (\mathcal{E} \cap [s, T_0]) \times A^{s, \omega} \in \mathcal{B}([s, T_0]) \otimes \mathcal{F}_{T_0}^s.$$

Hence, the rectangular measurable set $\mathcal{E} \times A \in \Lambda_{T_0} \triangleq \left\{ \mathcal{D} \subset [t, T_0] \times \Omega^t : \mathcal{D}^{s, \omega} \in \mathcal{B}([s, T_0]) \otimes \mathcal{F}_{T_0}^s \right\}$. In particular, $\emptyset \times \emptyset \in \Lambda_{T_0}$ and $[t, T_0] \times \Omega^t \in \Lambda_{T_0}$. For any $\mathcal{D} \in \Lambda_{T_0}$ and $\{\mathcal{D}_n\}_{n \in \mathbb{N}} \subset \Lambda_{T_0}$, similar to (4.5), we can deduce that

$(([t, T_0] \times \Omega^t) \setminus \mathcal{D})^{s, \omega} = (\mathcal{D}^{s, \omega})^c \in \mathcal{B}([s, T_0]) \otimes \mathcal{F}_{T_0}^s$, and that $\left(\bigcup_{n \in \mathbb{N}} \mathcal{D}_n\right)^{s, \omega} = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n^{s, \omega} \in \mathcal{B}([s, T_0]) \otimes \mathcal{F}_{T_0}^s$. Thus Λ_{T_0} is a σ -field of $[t, T_0] \times \Omega^t$. As the product σ -field $\mathcal{B}([t, T_0]) \otimes \mathcal{F}_{T_0}^t$ is generated by all rectangular measurable sets $\{\mathcal{E} \times A : \mathcal{E} \in \mathcal{B}([t, T_0]), A \in \mathcal{F}_{T_0}^t\}$, we can deduce that $\mathcal{B}([t, T_0]) \otimes \mathcal{F}_{T_0}^t \subset \Lambda_{T_0}$, i.e., $\mathcal{D}^{s, \omega} \in \mathcal{B}([s, T_0]) \otimes \mathcal{F}_{T_0}^s$ for any $\mathcal{D} \in \mathcal{B}([t, T_0]) \otimes \mathcal{F}_{T_0}^t$.

Let $\{X_r\}_{r \in [t, T]}$ be an \mathbb{M} -valued, measurable process on $(\Omega^t, \mathcal{F}_T^t)$. For any $\mathcal{M} \in \mathcal{B}(\mathbb{M})$, since $X^{-1}(\mathcal{M}) \in \mathcal{B}([t, T]) \otimes \mathcal{F}_T^t$, applying the above result with $T_0 = T$ yields that

$$\begin{aligned} (X^{s, \omega})^{-1}(\mathcal{M}) &= \{(r, \tilde{\omega}) \in [s, T] \times \Omega^s : X_r^{s, \omega}(\tilde{\omega}) \in \mathcal{M}\} = \{(r, \tilde{\omega}) \in [s, T] \times \Omega^s : X_r(\omega \otimes_s \tilde{\omega}) \in \mathcal{M}\} \\ &= \{(r, \tilde{\omega}) \in [s, T] \times \Omega^s : (r, \omega \otimes_s \tilde{\omega}) \in X^{-1}(\mathcal{M})\} = (X^{-1}(\mathcal{M}))^{s, \omega} \in \mathcal{B}([s, T]) \otimes \mathcal{F}_T^s, \end{aligned}$$

thus $\{X_r^{s, \omega}\}_{r \in [s, T]}$ is a measurable process on $(\Omega^s, \mathcal{F}_T^s)$.

Next, we consider an \mathbb{M} -valued, \mathbf{F}^t -progressively measurable process $\{X_r\}_{r \in [t, T]}$. For any $T_0 \in [s, T]$ and $\tilde{\mathcal{M}} \in \mathcal{B}(\mathbb{M})$, the \mathbf{F}^t -progressive measurability of X assures that $\tilde{\mathcal{D}} \triangleq \{(r, \omega') \in [t, T_0] \times \Omega^t : X_r(\omega') \in \tilde{\mathcal{M}}\} \in \mathcal{B}([t, T_0]) \otimes \mathcal{F}_{T_0}^t$, thus $\tilde{\mathcal{D}}^{s, \omega} \in \mathcal{B}([s, T_0]) \otimes \mathcal{F}_{T_0}^s$. It follows that

$$\begin{aligned} \{(r, \tilde{\omega}) \in [s, T_0] \times \Omega^s : X_r^{s, \omega}(\tilde{\omega}) \in \tilde{\mathcal{M}}\} &= \{(r, \tilde{\omega}) \in [s, T_0] \times \Omega^s : X_r(\omega \otimes_s \tilde{\omega}) \in \tilde{\mathcal{M}}\} \\ &= \{(r, \tilde{\omega}) \in [s, T_0] \times \Omega^s : (r, \omega \otimes_s \tilde{\omega}) \in \tilde{\mathcal{D}}\} = \tilde{\mathcal{D}}^{s, \omega} \in \mathcal{B}([s, T_0]) \otimes \mathcal{F}_{T_0}^s, \end{aligned}$$

which shows that $\{X_r^{s, \omega}\}_{r \in [s, T]}$ is \mathbf{F}^s -progressively measurable.

Moreover, for any $\mathcal{D} \in \mathcal{P}(\mathbf{F}^t)$, since $\mathbf{1}_{\mathcal{D}} = \{\mathbf{1}_{\mathcal{D}}(r, \omega') : r \in [t, T], \omega' \in \Omega^t\}$ is an \mathbf{F}^t -progressively measurable process, $(\mathbf{1}_{\mathcal{D}})^{s, \omega} = \mathbf{1}_{\mathcal{D}^{s, \omega}}$ is an \mathbf{F}^s -progressively measurable process, where we used the fact that

$$(\mathbf{1}_{\mathcal{D}})^{s, \omega}(r, \tilde{\omega}) = \mathbf{1}_{\mathcal{D}}(r, \omega \otimes_s \tilde{\omega}) = \mathbf{1}_{\mathcal{D}^{s, \omega}}(r, \tilde{\omega}), \quad \forall r \in [s, T], \tilde{\omega} \in \Omega^s.$$

Thus, $\mathcal{D}^{s, \omega} \in \mathcal{P}(\mathbf{F}^s)$. □

Proof of Corollary 4.1: Similar to the proof of Proposition 4.2, one can show that for any $\mathcal{D} \in \mathcal{P}(\mathbf{F}^t)$ and $\mathcal{M} \in \mathcal{B}(\mathbb{M})$, $(\mathcal{D} \times \mathcal{M})^{s, \omega} = \mathcal{D}^{s, \omega} \times \mathcal{M} \in \mathcal{P}(\mathbf{F}^s) \otimes \mathcal{B}(\mathbb{M})$, and that $\Lambda \triangleq \{\mathcal{J} \subset [t, T] \times \Omega^t \times \mathbb{M} : \mathcal{J}^{s, \omega} \in \mathcal{P}(\mathbf{F}^s) \otimes \mathcal{B}(\mathbb{M})\}$ forms a σ -field of $[t, T] \times \Omega^t \times \mathbb{M}$. Thus it follows that $\mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{M}) \subset \Lambda$, i.e., $\mathcal{J}^{s, \omega} \in \mathcal{P}(\mathbf{F}^s) \otimes \mathcal{B}(\mathbb{M})$ for any $\mathcal{J} \in \mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{M})$.

Next, let $f : [t, T] \times \Omega^t \times \mathbb{M} \rightarrow \tilde{\mathbb{M}}$ be a $\mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{M})/\mathcal{B}(\tilde{\mathbb{M}})$ -measurable function. For any $\mathcal{E} \in \mathcal{B}(\tilde{\mathbb{M}})$, the measurability of f assures that $\tilde{\mathcal{J}} \triangleq \{(r, \omega', x) \in [t, T] \times \Omega^t \times \mathbb{M} : f(r, \omega', x) \in \mathcal{E}\} \in \mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{M})$. Thus, $\tilde{\mathcal{J}}^{s, \omega} = \{(r, \tilde{\omega}, x) \in [s, T] \times \Omega^s \times \mathbb{M} : f^{s, \omega}(r, \tilde{\omega}, x) \in \mathcal{E}\} \in \mathcal{P}(\mathbf{F}^s) \otimes \mathcal{B}(\mathbb{M})$, which gives the measurability of $f^{s, \omega}$. □

Proof of Lemma 4.3: Set $A \triangleq \{\omega' \in \Omega^t : \tau(\omega') = \tau(\omega)\}$. For any $c \in \mathbb{R}^d$, $\xi^{-1}(c) = \{\omega' \in \Omega^t : \xi(\omega') = c\} \in \mathcal{F}_{\tau}^t$, thus $\xi^{-1}(c) \cap A \in \mathcal{F}_{\tau(\omega)}^t$. Since $\omega \in A = \bigcup_{c \in \mathbb{R}^d} (\xi^{-1}(c) \cap A)$, we can find some $\tilde{c} \in \mathbb{R}^d$ such that $\omega \in \xi^{-1}(\tilde{c}) \cap A \in \mathcal{F}_{\tau(\omega)}^t$. Then lemma 4.1 implies that $\omega \otimes_{\tau} \Omega^{\tau(\omega)} \subset \xi^{-1}(\tilde{c}) \cap A$. It follows that $\xi^{\tau, \omega}(\tilde{\omega}) = \xi(\omega \otimes_{\tau} \tilde{\omega}) = \xi(\omega) = \tilde{c}$ for any $\tilde{\omega} \in \Omega^{\tau(\omega)}$. It is clear that $\tau \in \mathcal{F}_{\tau}^t$, thus $\tau(\omega \otimes_{\tau} \tilde{\omega}) = \tau^{\tau, \omega}(\tilde{\omega}) = \tau(\omega)$ for any $\tilde{\omega} \in \Omega^{\tau(\omega)}$. □

Proof of Proposition 4.3: Fix $s \in [t, T]$. Let us first show that

$$E_s[\xi^{s, \omega}] = E_t[\xi | \mathcal{F}_s^t](\omega) \in \mathbb{R}, \quad \text{for } P_0^t - a.s. \omega \in \Omega^t. \quad (6.95)$$

In virtue of Theorem 1.3.4 and (1.3.15) of [45], P_0^t has a regular conditional probability distribution with respect to \mathcal{F}_s^t , i.e. a family $\{P_s^{\omega}\}_{\omega \in \Omega^t} \subset \mathcal{P}^t$ satisfying:

- (i) For any $A \in \mathcal{F}_T^t$, the mapping $\omega \rightarrow P_s^{\omega}(A)$ is \mathcal{F}_s^t -measurable;
- (ii) For any $\xi \in \mathbb{L}^1(\mathcal{F}_T^t)$, $E_{P_s^{\omega}}[\xi] = E_t[\xi | \mathcal{F}_s^t](\omega)$, for $P_0^t - a.s. \omega \in \Omega^t$; (6.96)

- (iii) For any $\omega \in \Omega^t$, $P_s^{\omega}(\omega \otimes_s \Omega^s) = 1$. (6.97)

Given $\omega \in \Omega^t$, since $\omega \otimes_s \tilde{A} \in \mathcal{F}_T^t$ for any $\tilde{A} \in \mathcal{F}_T^s$ by Lemma 4.2, one can deduce from (6.97) that

$$P^{s,\omega}(\tilde{A}) \triangleq P_s^\omega(\omega \otimes_s \tilde{A}), \quad \forall \tilde{A} \in \mathcal{F}_T^s$$

defines a probability measure on $(\Omega^s, \mathcal{F}_T^s)$.

For any $\tilde{A} \in \mathcal{F}_T^s$, (6.97) and (6.96) implies that for P_0^t -a.s. $\omega \in \Omega^t$

$$P^{s,\omega}(\tilde{A}) = P_s^\omega(\omega \otimes_s \tilde{A}) = P_s^\omega((\omega \otimes_s \Omega^s) \cap \Pi_{t,s}^{-1}(\tilde{A})) = P_s^\omega(\Pi_{t,s}^{-1}(\tilde{A})) = E_t[\mathbf{1}_{\Pi_{t,s}^{-1}(\tilde{A})} | \mathcal{F}_s^t](\omega).$$

It is easy to see that $\Pi_{t,s}^{-1}(\mathcal{F}_T^s) = \sigma(B_r^t - B_s^t; r \in [s, T])$. Thus $\Pi_{t,s}^{-1}(\tilde{A})$ is independent of \mathcal{F}_s^t . Applying Lemma 1.2 with $S = T$ yield that for P_0^t -a.s. $\omega \in \Omega^t$

$$P^{s,\omega}(\tilde{A}) = E_t[\mathbf{1}_{\Pi_{t,s}^{-1}(\tilde{A})} | \mathcal{F}_s^t](\omega) = E_t[\mathbf{1}_{\Pi_{t,s}^{-1}(\tilde{A})}] = P_0^t(\Pi_{t,s}^{-1}(\tilde{A})) = P_0^s(\tilde{A}).$$

Since \mathcal{C}_T^s is a countable set by Lemma 1.1, we can find a P_0^t -null set \mathcal{N} such that for any $\omega \in \mathcal{N}^c$, $P^{s,\omega}(\tilde{A}) = P_0^s(\tilde{A})$ holds for each $\tilde{A} \in \mathcal{C}_T^s \cup \{\Omega^s\}$. To wit, $\mathcal{C}_T^s \cup \{\Omega^s\} \subset \Lambda \triangleq \left\{ \tilde{A} \in \mathcal{F}_T^s : P^{s,\omega}(\tilde{A}) = P_0^s(\tilde{A}) \text{ for any } \omega \in \mathcal{N}^c \right\}$. It is easy to see that Λ is a Dynkin system. As \mathcal{C}_T^s is closed under intersection, Lemma 1.1 and Dynkin System Theorem show that $\mathcal{F}_T^s = \sigma(\mathcal{C}_T^s) \subset \Lambda$. Namely, it holds for any $\omega \in \mathcal{N}^c$ that

$$P^{s,\omega}(\tilde{A}) = P_0^s(\tilde{A}), \quad \forall \tilde{A} \in \mathcal{F}_T^s. \quad (6.98)$$

Let $A \in \mathcal{F}_T^t$. For any $\omega \in \Omega^t$, we have

$$(\mathbf{1}_A)^{s,\omega}(\tilde{\omega}) = \mathbf{1}_{\{\omega \otimes_s \tilde{\omega} \in A\}} = \mathbf{1}_{\{\tilde{\omega} \in A^{s,\omega}\}} = \mathbf{1}_{A^{s,\omega}}(\tilde{\omega}), \quad \forall \tilde{\omega} \in \Omega^s. \quad (6.99)$$

By (6.96), there exists a P_0^t -null set $\mathcal{N}(A)$ such that $P_s^\omega(A) = E_t[\mathbf{1}_A | \mathcal{F}_s^t](\omega) \in \mathbb{R}$. Given $\omega \in (\mathcal{N}^c \cup \mathcal{N}(A))^c$, since $A^{s,\omega} \in \mathcal{F}_T^s$ by Lemma 4.2, we see from (6.99), (6.98) and (6.97) that

$$\begin{aligned} E_s[(\mathbf{1}_A)^{s,\omega}] &= E_s[\mathbf{1}_{A^{s,\omega}}] = P_0^s(A^{s,\omega}) = P^{s,\omega}(A^{s,\omega}) = P_s^\omega(\omega \otimes_s A^{s,\omega}) \\ &= P_s^\omega((\omega \otimes_s \Omega^s) \cap A) = P_s^\omega(A) = E_t[\mathbf{1}_A | \mathcal{F}_s^t](\omega) \in \mathbb{R}. \end{aligned}$$

Then it follows that (6.95) holds for each simple \mathcal{F}_T^t -measurable random variable.

Now, for any $\xi \in \mathbb{L}^1(\mathcal{F}_T^t)$, ξ^+ can be approximated from below by a sequence of positive simple \mathcal{F}_T^t -measurable random variables: $\xi_n = \sum_{i=1}^{n^2-1} \frac{i}{n} \mathbf{1}_{A_n}$ with $A_n \triangleq \left\{ \xi^+ \in \left[\frac{i}{n}, \frac{i+1}{n} \right) \right\} \in \mathcal{F}_T^t$. It is clear that $\lim_{n \rightarrow \infty} \uparrow \xi_n^{s,\omega} = (\xi^+)^{s,\omega}$ for any $\omega \in \Omega^t$. Then Monotone Convergence Theorem implies that for P_0^t -a.s. $\omega \in \Omega^t$

$$E_s[(\xi^+)^{s,\omega}] = \lim_{n \rightarrow \infty} \uparrow E_s[\xi_n^{s,\omega}] = \lim_{n \rightarrow \infty} \uparrow E_t[\xi_n | \mathcal{F}_s^t](\omega) = E_t[\xi^+ | \mathcal{F}_s^t](\omega) < \infty. \quad (6.100)$$

Similarly, $E_s[(\xi^-)^{s,\omega}] = E_t[\xi^- | \mathcal{F}_s^t](\omega) < \infty$ for P_0^t -a.s. $\omega \in \Omega^t$, which together with (6.100) shows that (6.95) holds for both $|\xi|$ and ξ . Let τ take values in $\{t_n\}_{n \in \mathbb{N}} \subset [t, T]$, it then follows that for P_0^t -a.s. $\omega \in \Omega^t$

$$\begin{aligned} E_{\tau(\omega)}[|\xi^{\tau,\omega}|] &= E_{\tau(\omega)}[|\xi|^{\tau,\omega}] = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{\tau(\omega)=t_n\}} E_{t_n}[|\xi|^{t_n,\omega}] = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{\tau(\omega)=t_n\}} E_t[|\xi| | \mathcal{F}_{t_n}^t](\omega) \\ &= \sum_{n \in \mathbb{N}} \mathbf{1}_{\{\tau(\omega)=t_n\}} E_t[|\xi| | \mathcal{F}_\tau^t](\omega) = E_t[|\xi| | \mathcal{F}_\tau^t](\omega) \in [0, \infty) \end{aligned}$$

Similarly, we see that (4.6) holds for P_0^t -a.s. $\omega \in \Omega^t$. \square

Proof of Corollary 4.2: Given a P_0^t -null set \mathcal{N} , there exists an $A \in \mathcal{F}_T^t$ with $P_0^t(A) = 0$ such that $\mathcal{N} \subset A$. For any $\omega \in \Omega^t$, applying (6.99) with $s = \tau(\omega)$ gives that $(\mathbf{1}_A)^{\tau,\omega} = \mathbf{1}_{A^{\tau,\omega}}$; Also, (4.3) and Lemma 4.2 show that $\mathcal{N}^{\tau,\omega} \subset A^{\tau,\omega} \in \mathcal{F}_T^{\tau(\omega)}$. Then (4.6) imply that for P_0^t -a.s. $\omega \in \Omega^t$

$$P_0^{\tau(\omega)}(A^{\tau,\omega}) = E_{\tau(\omega)}[\mathbf{1}_{A^{\tau,\omega}}] = E_{\tau(\omega)}[(\mathbf{1}_A)^{\tau,\omega}] = E_t[\mathbf{1}_A | \mathcal{F}_\tau^t](\omega) = 0,$$

thus $\mathcal{N}^{\tau, \omega} \in \mathcal{N}^{P_0^{\tau(\omega)}}$. Next let ξ_1 and ξ_2 be two real-valued random variables with $\xi_1 \leq \xi_2$, P_0^t -a.s. Since $\mathcal{N} \triangleq \{\omega \in \Omega^t : \xi_1(\omega) > \xi_2(\omega)\} \in \mathcal{N}^{P_0^t}$, it holds for P_0^t -a.s. $\omega \in \Omega^t$ that

$$0 = P_0^{\tau(\omega)}(\mathcal{N}^{\tau, \omega}) = P_0^{\tau(\omega)}\{\tilde{\omega} \in \Omega^{\tau(\omega)} : \xi_1(\omega \otimes_{\tau} \tilde{\omega}) > \xi_2(\omega \otimes_{\tau} \tilde{\omega})\} = P_0^{\tau(\omega)}\{\tilde{\omega} \in \Omega^{\tau(\omega)} : \xi_1^{\tau, \omega}(\tilde{\omega}) > \xi_2^{\tau, \omega}(\tilde{\omega})\}. \quad \square$$

Proof of Proposition 4.4: For each $\omega \in \Omega^t$, applying Proposition 4.2 with $s = \tau(\omega)$ and $T_0 = T$ shows that $\{X_r^{\tau, \omega}\}_{r \in [\tau(\omega), T]}$ is a measurable process on $(\Omega^{\tau(\omega)}, \mathcal{F}_T^{\tau(\omega)})$. Since $E_t\left[\left(\int_{\tau}^T |X_r|^p dr\right)^{\hat{p}/p}\right] < \infty$, the integral $\int_t^T \mathbf{1}_{\{r \geq \tau(\omega)\}} |X_r(\omega)|^p dr$ is well-defined for all $\omega \in \Omega^t$ except on an P_0^t -null set \mathcal{N} . Let $\xi \triangleq \mathbf{1}_{\mathcal{N}^c} \int_{\tau}^T |X_r|^p dr$, so $\xi^{\hat{p}/p} \in \mathbb{L}^1(\mathcal{F}_T^t)$. Given $\omega \in \Omega^t$, (4.2) and Lemma 4.3 show that

$$\xi^{\tau, \omega}(\tilde{\omega}) = \mathbf{1}_{(\mathcal{N}^c)^{\tau, \omega}}(\tilde{\omega}) \cdot \int_{\tau(\omega \otimes_{\tau} \tilde{\omega})}^T |X_r(\omega \otimes_{\tau} \tilde{\omega})|^p dr = \mathbf{1}_{(\mathcal{N}^{\tau, \omega})^c}(\tilde{\omega}) \cdot \int_{\tau(\omega)}^T |X_r^{\tau, \omega}(\tilde{\omega})|^p dr, \quad \forall \tilde{\omega} \in \Omega^{\tau(\omega)}. \quad (6.101)$$

By Corollary 4.2, it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\mathcal{N}^{\tau, \omega} \in \mathcal{N}^{P_0^{\tau(\omega)}}$. Then (6.101) and (4.6) imply that for P_0^t -a.s. $\omega \in \Omega^t$

$$E_{\tau(\omega)}\left[\left(\int_{\tau(\omega)}^T |X_r^{\tau, \omega}|^p dr\right)^{\hat{p}/p}\right] = E_{\tau(\omega)}\left[(\xi^{\hat{p}/p})^{\tau, \omega}\right] = E_t\left[\xi^{\hat{p}/p} | \mathcal{F}_T^t\right](\omega) = E_t\left[\left(\int_{\tau}^T |X_r|^p dr\right)^{\hat{p}/p} | \mathcal{F}_T^t\right](\omega) < \infty. \quad \square$$

Proof of Corollary 4.3: Firstly, let $\{X_r\}_{r \in [t, T]} \in \mathbb{H}_{\mathbf{F}^t}^{p, \hat{p}}([t, T], \mathbb{E})$. For each $\omega \in \Omega^t$, applying Proposition 4.2 with $s = \tau(\omega)$ shows that $\{X_r^{\tau, \omega}\}_{r \in [\tau(\omega), T]}$ is $\mathbf{F}^{\tau(\omega)}$ -progressively measurable. Then Proposition 4.4 implies that for P_0^t -a.s. $\omega \in \Omega^t$, $\{X_r^{\tau, \omega}\}_{r \in [\tau(\omega), T]} \in \mathbb{H}_{\mathbf{F}^{\tau(\omega)}}^{p, \hat{p}}([\tau(\omega), T], \mathbb{E}, P_0^{\tau(\omega)})$.

Next, let $\{X_r\}_{r \in [t, T]} \in \mathbb{C}_{\mathbf{F}^t}^p([t, T], \mathbb{E})$ with continuous paths except on an P_0^t -null set \mathcal{N} . Given $\omega \in \Omega^t$, Proposition 4.1 shows that $\{X_r^{\tau, \omega}\}_{r \in [\tau(\omega), T]}$ is $\mathbf{F}^{\tau(\omega)}$ -adapted, and the path $[\tau(\omega), T] \ni r \rightarrow X_r^{\tau, \omega}(\tilde{\omega}) = X_r(\omega \otimes_{\tau} \tilde{\omega})$ is continuous for any $\tilde{\omega} \in (\mathcal{N}^c)^{\tau, \omega}$. By (4.2) and Corollary 4.2, it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $P_0^{\tau(\omega)}((\mathcal{N}^c)^{\tau, \omega}) = P_0^{\tau(\omega)}((\mathcal{N}^{\tau, \omega})^c) = 1$. Moreover, applying Proposition 4.3 with $\xi = \sup_{r \in [t, T]} |X_r| \in \mathbb{L}^p(\mathcal{F}_T^t)$ yields that for P_0^t -a.s. $\omega \in \Omega^t$,

$$\Omega^t, E_{\tau(\omega)}\left[\sup_{r \in [\tau(\omega), T]} |X_r^{\tau, \omega}|^p\right] \leq E_{\tau(\omega)}[|\xi^{\tau, \omega}|^p] < \infty. \quad \square$$

Proof of Proposition 4.5: We set $\mathcal{D}_r \triangleq \{\omega \in \Omega^t : (r, \omega) \in \mathcal{D}\}$, $\forall r \in [t, T]$. Fubini Theorem shows that

$$0 = (dr \times dP_0^t)(\mathcal{D} \cap [\tau, T]) = \int_{\Omega^t} \left(\int_{\tau(\omega)}^T \mathbf{1}_{\mathcal{D}_r}(\omega) dr\right) dP_0^t(\omega) = E_t\left[\int_{\tau}^T \mathbf{1}_{\mathcal{D}_r} dr\right].$$

Thus $\int_{\tau}^T \mathbf{1}_{\mathcal{D}_r} dr \in \mathbb{L}^1(\mathcal{F}_T^t)$ is equal to 0, P_0^t -a.s., which together with (4.6) implies that

$$E_{\tau(\omega)}\left[\left(\int_{\tau}^T \mathbf{1}_{\mathcal{D}_r} dr\right)^{\tau, \omega}\right] = E_t\left[\int_{\tau}^T \mathbf{1}_{\mathcal{D}_r} dr | \mathcal{F}_T^t\right](\omega) = 0. \quad (6.102)$$

holds for any $\omega \in \Omega^t$ except on an P_0^t -null set \mathcal{N} . Given $\omega \in \mathcal{N}^c$, applying Proposition 4.2 with $s = \tau(\omega)$ and $T_0 = T$ yields that $\mathcal{D}^{\tau, \omega} \in \mathcal{B}([\tau(\omega), T]) \otimes \mathcal{F}_T^{\tau(\omega)}$. Since

$$(\mathcal{D}^{\tau, \omega})_r = \{\tilde{\omega} \in \Omega^{\tau(\omega)} : (r, \tilde{\omega}) \in \mathcal{D}^{\tau, \omega}\} = \{\tilde{\omega} \in \Omega^{\tau(\omega)} : (r, \omega \otimes_{\tau} \tilde{\omega}) \in \mathcal{D}\} = \{\tilde{\omega} \in \Omega^{\tau(\omega)} : \omega \otimes_{\tau} \tilde{\omega} \in \mathcal{D}_r\}$$

for any $r \in [\tau(\omega), T]$, we can deduce from Fubini Theorem, Lemma 4.3 and (6.102) that

$$\begin{aligned} (dr \times dP_0^{\tau(\omega)})(\mathcal{D}^{\tau, \omega}) &= \int_{\Omega^{\tau(\omega)}} \left(\int_{\tau(\omega)}^T \mathbf{1}_{\mathcal{D}_r^{\tau, \omega}}(\tilde{\omega}) dr\right) dP_0^{\tau(\omega)}(\tilde{\omega}) = \int_{\Omega^{\tau(\omega)}} \left(\int_{\tau(\omega \otimes_{\tau} \tilde{\omega})}^T \mathbf{1}_{\mathcal{D}_r}(\omega \otimes_{\tau} \tilde{\omega}) dr\right) dP_0^{\tau(\omega)}(\tilde{\omega}) \\ &= \int_{\Omega^{\tau(\omega)}} \left(\int_{\tau}^T \mathbf{1}_{\mathcal{D}_r} dr\right)^{\tau, \omega}(\tilde{\omega}) dP_0^{\tau(\omega)}(\tilde{\omega}) = E_{\tau(\omega)}\left[\left(\int_{\tau}^T \mathbf{1}_{\mathcal{D}_r} dr\right)^{\tau, \omega}\right] = 0. \end{aligned} \quad \square$$

Proof of Proposition 4.6: (1) Let $\mu \in \mathcal{U}^t$. Given $\omega \in \Omega^t$, we see from Proposition 4.2 that $\mu^{\tau, \omega} = \{\mu_r^{\tau, \omega}\}_{r \in [\tau(\omega), T]}$ is an $\mathbf{F}^{\tau(\omega)}$ -progressively measurable process, and Lemma 4.3 shows that

$$\left(\int_{\tau}^T [\mu_r]_{\mathbb{U}}^2 dr \right)^{\tau, \omega}(\tilde{\omega}) = \int_{\tau(\omega \otimes_{\tau} \tilde{\omega})}^T [\mu_r(\omega \otimes_{\tau} \tilde{\omega})]_{\mathbb{U}}^2 dr = \int_{\tau(\omega)}^T [\mu_r^{\tau, \omega}(\tilde{\omega})]_{\mathbb{U}}^2 dr, \quad \forall \tilde{\omega} \in \Omega^{\tau(\omega)}. \quad (6.103)$$

As $\mathcal{D} \triangleq \{(r, \omega) \in [t, T] \times \Omega^t : \mu_r(\omega) \in \mathbb{U} \setminus \mathbb{U}_0\}$ has zero $dr \times dP_0^t$ -measure, we see from Proposition 4.5 that for P_0^t -a.s. $\omega \in \Omega^t$,

$$0 = (dr \times dP_0^{\tau(\omega)})(\mathcal{D}^{\tau, \omega}) = (dr \times dP_0^{\tau(\omega)})(\{(r, \tilde{\omega}) \in [\tau(\omega), T] \times \Omega^{\tau(\omega)} : \mu_r^{\tau, \omega}(\tilde{\omega}) \in \mathbb{U} \setminus \mathbb{U}_0\}).$$

On the other hand, applying Proposition 4.3 with $\xi = \int_{\tau}^T [\mu_r]_{\mathbb{U}}^2 dr \in \mathbb{L}^1(\mathcal{F}_T^t)$ and using (6.103) yield that for P_0^t -a.s. $\omega \in \Omega^t$,

$$E_{\tau(\omega)} \left[\int_{\tau(\omega)}^T [\mu_r^{\tau, \omega}]_{\mathbb{U}}^2 dr \right] = E_t \left[\int_{\tau}^T [\mu_r]_{\mathbb{U}}^2 dr \middle| \mathcal{F}_{\tau}^t \right](\omega) \leq E_t \left[\int_t^T [\mu_r]_{\mathbb{U}}^2 dr \middle| \mathcal{F}_{\tau}^t \right](\omega) < \infty, \quad (6.104)$$

thus $\mu^{\tau, \omega} \in \mathcal{U}^{\tau(\omega)}$. Similarly, one can show that for any $\nu \in \mathcal{V}^t$, it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\nu^{\tau, \omega} \in \mathcal{V}^{\tau(\omega)}$.

(2) Let $\alpha \in \mathcal{A}^t$. There exists a $dr \times dP_0^t$ -a.s. null set $\mathcal{D} \subset [t, T] \times \Omega^t$ such that $\alpha(r, \omega, \mathbb{V}_0) \subset \mathbb{U}_0$ and (2.25) holds for all $(r, \omega) \in ([s, T] \times \Omega^t) \setminus \mathcal{D}$. Given $\omega \in \Omega^t$, Corollary 4.1 shows that $\alpha^{s, \omega}(r, \tilde{\omega}, v) = \alpha(r, \omega \otimes_s \tilde{\omega}, v)$, $\forall (r, \tilde{\omega}, v) \in [s, T] \times \Omega^s \times \mathbb{V}$ is $\mathcal{P}(\mathbf{F}^s) \otimes \mathcal{B}(\mathbb{V})/\mathcal{B}(\mathbb{U})$ -measurable, and we can deduce that for any $(r, \tilde{\omega}) \in ([s, T] \times \Omega^s) \setminus \mathcal{D}^{s, \omega}$ (or equivalent, $(r, \omega \otimes_s \tilde{\omega}) \in ([s, T] \times \Omega^t) \setminus \mathcal{D}$), $\alpha^{s, \omega}(r, \tilde{\omega}, \mathbb{V}_0) = \alpha(r, \omega \otimes_s \tilde{\omega}, \mathbb{V}_0) \subset \mathbb{U}_0$ and

$$[\alpha^{s, \omega}(r, \tilde{\omega}, v)]_{\mathbb{U}} = [\alpha(r, \omega \otimes_s \tilde{\omega}, v)]_{\mathbb{U}} \leq \Psi_r(\omega \otimes_s \tilde{\omega}) + \kappa[v]_{\mathbb{V}} = \Psi_r^{s, \omega}(\tilde{\omega}) + \kappa[v]_{\mathbb{V}}, \quad \forall v \in \mathbb{V}. \quad (6.105)$$

In light of Proposition 4.5 and Proposition 4.4, it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\mathcal{D}^{s, \omega}$ is a $dr \times dP_0^s$ -a.s. null set and that $\Psi^{s, \omega}$ is a non-negative measurable process on $(\Omega^s, \mathcal{F}_T^s)$ with $E_s \int_s^T (\Psi_r^{s, \omega})^2 dr < \infty$. Hence, $\alpha^{s, \omega} \in \mathcal{A}^s$ for P_0^t -a.s. $\omega \in \Omega^t$.

Moreover, assuming that α additionally satisfies (2.26), i.e. $\alpha \in \hat{\mathcal{A}}^t$, we shall show that $\alpha^{s, \omega}$ also satisfies (2.26) for P_0^t -a.s. $\omega \in \Omega^t$. Given $n \in \mathbb{N}$, there exist a δ_n and a closed subset F_n of Ω^t with $P_0^t(F_n) > 1 - \frac{1}{n}$ such that for any $\omega, \omega' \in F_n$ with $\|\omega - \omega'\|_t < \delta_n$

$$\sup_{r \in [t, T]} \sup_{v \in \mathbb{V}} \rho_{\mathbb{U}}(\alpha(r, \omega, v), \alpha(r, \omega', v)) < \frac{1}{n}. \quad (6.106)$$

For any $\omega \in \Omega^t$, Lemma 4.2 shows that $F_n^{s, \omega}$ is a closed subset of Ω^s . Applying Proposition 4.3 with $\xi = \mathbf{1}_{F_n}$ and using (6.99) show that for all $\omega \in \Omega^t$ except on a P_0^t -null set \mathcal{N}_n

$$P_0^s(F_n^{s, \omega}) = E_s[\mathbf{1}_{F_n^{s, \omega}}] = E_s[(\mathbf{1}_{F_n})^{s, \omega}] = E_t[\mathbf{1}_{F_n} | \mathcal{F}_s^t](\omega) < \infty. \quad (6.107)$$

As $\lim_{n \rightarrow \infty} E_t[1 - E_t[\mathbf{1}_{F_n} | \mathcal{F}_s^t]] = 1 - \lim_{n \rightarrow \infty} P_0^t(F_n) = 0$, there exists a subsequence $\{n_i\}_{i \in \mathbb{N}}$ of \mathbb{N} such that $\lim_{i \rightarrow \infty} E_t[\mathbf{1}_{F_{n_i}} | \mathcal{F}_s^t] = 1$ holds except on a P_0^t -null set \mathcal{N} . Thus, taking $n = n_i$ in (6.107) and letting $i \rightarrow \infty$ yield that $\lim_{i \rightarrow \infty} P_0^s(F_{n_i}^{s, \omega}) = 1$

for all $\omega \in \Omega^t$ except on the P_0^t -null set $\tilde{\mathcal{N}} \triangleq (\bigcup_{i \in \mathbb{N}} \mathcal{N}_{n_i}) \cup \mathcal{N}$.

Now, fix $\omega \in \tilde{\mathcal{N}}^c$. For any $\varepsilon > 0$, there exists an $i \in \mathbb{N}$ such that $n_i > \frac{1}{\varepsilon}$ and that $P_0^s(F_{n_i}^{s, \omega}) > 1 - \varepsilon$. Let $\tilde{\omega} \in F_{n_i}^{s, \omega}$. For any $\tilde{\omega}' \in F_{n_i}^{s, \omega}$ with $\|\tilde{\omega} - \tilde{\omega}'\|_s < \delta_{n_i}$, since $\sup_{r \in [t, T]} |(\omega \otimes_s \tilde{\omega})(r) - (\omega \otimes_s \tilde{\omega}')(r)| = \sup_{r \in [s, T]} |\tilde{\omega}(r) - \tilde{\omega}'(r)| < \delta_{n_i}$, we see from (6.106) that

$$\sup_{r \in [t, T]} \sup_{v \in \mathbb{V}} \rho_{\mathbb{U}}(\alpha^{s, \omega}(r, \tilde{\omega}, v), \alpha^{s, \omega}(r, \tilde{\omega}', v)) = \sup_{r \in [t, T]} \sup_{v \in \mathbb{V}} \rho_{\mathbb{U}}(\alpha(r, \omega \otimes_s \tilde{\omega}, v), \alpha(r, \omega \otimes_s \tilde{\omega}', v)) < \frac{1}{n_i} < \varepsilon.$$

Hence, $\alpha^{s, \omega}$ satisfies (2.26).

Similarly, one can show that for any $\beta \in \mathfrak{B}^t$ (resp. $\beta \in \hat{\mathfrak{B}}^t$), it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $\beta^{s, \omega} \in \mathfrak{B}^s$ (resp. $\beta^{s, \omega} \in \hat{\mathfrak{B}}^s$). \square

Proof of Lemma 4.4: Let $\{\xi_i\}_{i \in \mathbb{N}}$ be a sequence of $L^1(\mathcal{F}_T^t)$ that converges to 0 in probability P_0^t , i.e.

$$\lim_{i \rightarrow \infty} \downarrow E_t[\mathbf{1}_{\{|\xi_i| > 1/n\}}] = \lim_{i \rightarrow \infty} \downarrow P_0^t(|\xi_i| > 1/n) = 0, \quad \forall n \in \mathbb{N}. \quad (6.108)$$

In particular, $\lim_{i \rightarrow \infty} \downarrow E_t[\mathbf{1}_{\{|\xi_i| > 1\}}] = 0$ allows us to extract a subsequence $S_1 = \{\xi_i^1\}_{i \in \mathbb{N}}$ from $\{\xi_i\}_{i \in \mathbb{N}}$ such that $\lim_{i \rightarrow \infty} \mathbf{1}_{\{|\xi_i^1| > 1\}} = 0$, P_0^t -a.s. Clearly, S_1 also satisfies (6.108). Then by $\lim_{i \rightarrow \infty} \downarrow E_t[\mathbf{1}_{\{|\xi_i^1| > 1/2\}}] = 0$, we can find a subsequence $S_2 = \{\xi_i^2\}_{i \in \mathbb{N}}$ of S_1 such that $\lim_{i \rightarrow \infty} \mathbf{1}_{\{|\xi_i^2| > 1/2\}} = 0$, P_0^t -a.s. Inductively, for each $n \in \mathbb{N}$ we can select a subsequence $S_{n+1} = \{\xi_i^{n+1}\}_{i \in \mathbb{N}}$ of $S_n = \{\xi_i^n\}_{i \in \mathbb{N}}$ such that $\lim_{i \rightarrow \infty} \mathbf{1}_{\{|\xi_i^{n+1}| > \frac{1}{n+1}\}} = 0$, P_0^t -a.s.

For any $i \in \mathbb{N}$, we set $\tilde{\xi}_i \triangleq \xi_i^i$, which belongs to S_n for $n = 1, \dots, i$. Given $n \in \mathbb{N}$, since $\{\tilde{\xi}_i\}_{i=n}^\infty \subset S_n$, it holds P_0^t -a.s. that $\lim_{i \rightarrow \infty} \mathbf{1}_{\{|\tilde{\xi}_i| > \frac{1}{n}\}} = 0$. Then Bound Convergence Theorem and Proposition 4.3 imply that

$$0 = \lim_{i \rightarrow \infty} E_t[\mathbf{1}_{\{|\tilde{\xi}_i| > 1/n\}} | \mathcal{F}_\tau^t](\omega) = \lim_{i \rightarrow \infty} E_{\tau(\omega)}[(\mathbf{1}_{\{|\tilde{\xi}_i| > 1/n\}})^{\tau, \omega}] \quad (6.109)$$

holds for all $\omega \in \Omega^t$ except on a P_0^t -null set \mathcal{N}_n . Let $\omega \in \left(\bigcup_{n \in \mathbb{N}} \mathcal{N}_n\right)^c$. For any $n \in \mathbb{N}$, one can deduce that

$$(\mathbf{1}_{\{|\tilde{\xi}_i| > 1/n\}})^{\tau, \omega}(\tilde{\omega}) = \mathbf{1}_{\{|\tilde{\xi}_i(\omega \otimes_\tau \tilde{\omega})| > 1/n\}} = \mathbf{1}_{\{|\tilde{\xi}_i^{\tau, \omega}(\tilde{\omega})| > 1/n\}} = (\mathbf{1}_{\{|\tilde{\xi}_i^{\tau, \omega}| > 1/n\}})^{\tau, \omega}(\tilde{\omega}), \quad \forall \tilde{\omega} \in \Omega^{\tau(\omega)}.$$

which together with (6.109) leads to that $\lim_{i \rightarrow \infty} P_0^{\tau(\omega)}(|\tilde{\xi}_i^{\tau, \omega}| > 1/n) = \lim_{i \rightarrow \infty} E_{\tau(\omega)}[(\mathbf{1}_{\{|\tilde{\xi}_i| > 1/n\}})^{\tau, \omega}] = 0$. \square

Proof of Proposition 4.7: Since $\tilde{X}^\Theta \in \mathbb{C}_{\mathbf{F}^t}^2([t, T], \mathbb{R}^k)$, Corollary 4.3 shows that for P_0^t -a.s. $\omega \in \Omega^t$, $\{(\tilde{X}^\Theta)_r^{\tau, \omega}\}_{r \in [\tau(\omega), T]} \in \mathbb{C}_{\mathbf{F}^{\tau(\omega)}}^2([\tau(\omega), T], \mathbb{R}^k, P_0^{\tau(\omega)})$. Thus, it suffices to show that for P_0^t -a.s. $\omega \in \Omega^t$, $(\tilde{X}^\Theta)^{\tau, \omega}$ solves (4.7).

As the \mathbf{F}^t -version of X^Θ , \tilde{X}^Θ also satisfies (1.1). Thus, one can deduce that except on a P_0^t -null set \mathcal{N}

$$\tilde{X}_{\tau \vee s}^\Theta - \tilde{X}_\tau^\Theta = \int_{\tau \vee s}^\tau b(r, \tilde{X}_r^\Theta, \mu_r, \nu_r) dr + \int_{\tau \vee s}^\tau \sigma(r, \tilde{X}_r^\Theta, \mu_r, \nu_r) dB_r^t = \int_t^s b_r^\Theta dr + \int_t^s \sigma_r^\Theta dB_r^t, \quad s \in [t, T], \quad (6.110)$$

where $b_r^\Theta \triangleq \mathbf{1}_{\{r > \tau\}} b(r, \tilde{X}_r^\Theta, \mu_r, \nu_r)$ and $\sigma_r^\Theta \triangleq \mathbf{1}_{\{r > \tau\}} \sigma(r, \tilde{X}_r^\Theta, \mu_r, \nu_r)$. In light of (2.2), (2.6) and (2.7), $M_s^\Theta \triangleq \int_t^s \sigma_r^\Theta dB_r^t$, $s \in [t, T]$ is a square-integrable martingale with respect to $(\bar{\mathbf{F}}^t, P_0^t)$.

Since $\tilde{X}_\tau^\Theta \in \mathbf{F}_\tau^t$, Lemma 4.3 shows that for any $\omega \in \Omega^t$ and $\tilde{\omega} \in \Omega^{\tau(\omega)}$

$$(\tilde{X}_\tau^\Theta)(\omega \otimes_\tau \tilde{\omega}) = (\tilde{X}_\tau^\Theta)(\omega) = \tilde{X}_{\tau(\omega)}^\Theta(\omega) \quad \text{and} \quad \tau(\omega \otimes_\tau \tilde{\omega}) = \tau(\omega). \quad (6.111)$$

Fix $\omega \in \Omega^t$. It easily follows that for any $\tilde{\omega} \in \Omega^{\tau(\omega)}$

$$(\tilde{X}_{\tau \vee s}^\Theta)(\omega \otimes_\tau \tilde{\omega}) = \tilde{X}_{\tau(\omega \otimes_\tau \tilde{\omega}) \vee s}^\Theta(\omega \otimes_\tau \tilde{\omega}) = \tilde{X}_{\tau(\omega) \vee s}^\Theta(\omega \otimes_\tau \tilde{\omega}) = \tilde{X}_s^\Theta(\omega \otimes_\tau \tilde{\omega}) = (\tilde{X}^\Theta)_s^{\tau, \omega}(\tilde{\omega}), \quad \forall s \in [\tau(\omega), T], \quad (6.112)$$

$$\begin{aligned} b_r^\Theta(\omega \otimes_\tau \tilde{\omega}) &= \mathbf{1}_{\{r > \tau(\omega \otimes_\tau \tilde{\omega})\}} b(r, \tilde{X}_r^\Theta(\omega \otimes_\tau \tilde{\omega}), \mu_r(\omega \otimes_\tau \tilde{\omega}), \nu_r(\omega \otimes_\tau \tilde{\omega})) \\ &= \mathbf{1}_{\{r > \tau(\omega)\}} b(r, (\tilde{X}^\Theta)_r^{\tau, \omega}(\tilde{\omega}), \mu_r^{\tau, \omega}(\tilde{\omega}), \nu_r^{\tau, \omega}(\tilde{\omega})), \quad \forall r \in [t, T], \end{aligned} \quad (6.113)$$

and similarly

$$\sigma_r^\Theta(\omega \otimes_\tau \tilde{\omega}) = \mathbf{1}_{\{r > \tau(\omega)\}} \sigma(r, (\tilde{X}^\Theta)_r^{\tau, \omega}(\tilde{\omega}), \mu_r^{\tau, \omega}(\tilde{\omega}), \nu_r^{\tau, \omega}(\tilde{\omega})), \quad \forall r \in [t, T]. \quad (6.114)$$

Then for any $\tilde{\omega} \in (\mathcal{N}^c)^{\tau, \omega}$, applying (6.110) to the path $\omega \otimes_\tau \tilde{\omega}$ over period $[\tau(\omega), T]$ and using (6.111)-(6.113) yield

$$(\tilde{X}^\Theta)_s^{\tau, \omega}(\tilde{\omega}) = \tilde{X}_{\tau(\omega)}^\Theta(\omega) + \int_{\tau(\omega)}^s b(r, (\tilde{X}^\Theta)_r^{\tau, \omega}(\tilde{\omega}), \mu_r^{\tau, \omega}(\tilde{\omega}), \nu_r^{\tau, \omega}(\tilde{\omega})) dr + (M^\Theta)_s^{\tau, \omega}(\tilde{\omega}), \quad s \in [\tau(\omega), T].$$

By (4.2) and Corollary 4.2, it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $P_0^{\tau(\omega)}((\mathcal{N}^c)^{\tau, \omega}) = P_0^{\tau(\omega)}((\mathcal{N}^{\tau, \omega})^c) = 1$. Hence, it remains to show that for P_0^t -a.s. $\omega \in \Omega^t$, it holds $P_0^{\tau(\omega)}$ -a.s. that

$$(M^\Theta)_s^{\tau, \omega} = \int_{\tau(\omega)}^s \sigma(r, (\tilde{X}^\Theta)_r^{\tau, \omega}, \mu_r^{\tau, \omega}, \nu_r^{\tau, \omega}) dB_r^{\tau(\omega)}, \quad s \in [\tau(\omega), T].$$

Since M^Θ is a square-integrable martingale with respect to $(\bar{\mathbf{F}}^t, P_0^t)$, we know that (see e.g. Problem 3.2.27 of [27]) there is a sequence of $\mathbb{R}^{k \times d}$ -valued, \mathbf{F}^t -simple processes $\left\{ \Phi_s^n = \sum_{i=1}^{\ell_n} \xi_i^n \mathbf{1}_{\{s \in (t_i^n, t_{i+1}^n]\}} \right\}_{n \in \mathbb{N}}$ (where $t = t_1^n < \dots < t_{\ell_n+1}^n = T$ and $\xi_i^n \in \mathcal{F}_{t_i^n}^t$ for $i = 1, \dots, \ell_n$) such that

$$P_0^t - \lim_{n \rightarrow \infty} \int_t^T \text{trace} \left\{ (\Phi_r^n - \sigma_r^\Theta) (\Phi_r^n - \sigma_r^\Theta)^T \right\} ds = 0 \quad \text{and} \quad P_0^t - \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} |M_s^n - M_s^\Theta| = 0.$$

where $M_s^n \triangleq \int_t^s \Phi_r^n dB_r^t = \sum_{i=1}^{\ell_n} \xi_i^n (B_{s \wedge t_{i+1}^n}^t - B_{s \wedge t_i^n}^t)$. Then it directly follows that

$$P_0^t - \lim_{n \rightarrow \infty} \int_\tau^T \text{trace} \left\{ (\Phi_r^n - \sigma_r^\Theta) (\Phi_r^n - \sigma_r^\Theta)^T \right\} ds = 0 \quad \text{and} \quad P_0^t - \lim_{n \rightarrow \infty} \sup_{s \in [\tau, T]} |M_s^n - M_s^\Theta| = 0.$$

By Lemma 4.4, $\{\Phi^n\}_{n \in \mathbb{N}}$ has a subsequence $\left\{ \widehat{\Phi}_s^n = \sum_{i=1}^{\widehat{\ell}_n} \widehat{\xi}_i^n \mathbf{1}_{\{s \in (\widehat{t}_i^n, \widehat{t}_{i+1}^n]\}} \right\}_{n \in \mathbb{N}}$ such that except on a P_0^t -null set $\widehat{\mathcal{N}}$

$$\begin{aligned} 0 &= P_0^{\tau(\omega)} - \lim_{n \rightarrow \infty} \left(\int_\tau^T \text{trace} \left\{ (\widehat{\Phi}_r^n - \sigma_r^\Theta) (\widehat{\Phi}_r^n - \sigma_r^\Theta)^T \right\} ds \right)^{\tau, \omega} \\ &= P_0^{\tau(\omega)} - \lim_{n \rightarrow \infty} \int_{\tau(\omega)}^T \text{trace} \left\{ \left((\widehat{\Phi}_r^n)^{\tau, \omega} - (\sigma_r^\Theta)^{\tau, \omega} \right) \left((\widehat{\Phi}_r^n)^{\tau, \omega} - (\sigma_r^\Theta)^{\tau, \omega} \right)^T \right\} ds \end{aligned} \quad (6.115)$$

$$\text{and } 0 = P_0^{\tau(\omega)} - \lim_{n \rightarrow \infty} \left(\sup_{s \in [\tau, T]} |\widehat{M}_s^n - M_s^\Theta| \right)^{\tau, \omega} = P_0^{\tau(\omega)} - \lim_{n \rightarrow \infty} \sup_{s \in [\tau(\omega), T]} \left| (\widehat{M}_s^n)^{\tau, \omega} - (M_s^\Theta)^{\tau, \omega} \right|, \quad (6.116)$$

where $\widehat{M}_s^n \triangleq \int_t^s \widehat{\Phi}_r^n dB_r^t = \sum_{i=1}^{\widehat{\ell}_n} \widehat{\xi}_i^n (B_{s \wedge \widehat{t}_{i+1}^n}^t - B_{s \wedge \widehat{t}_i^n}^t)$ and we made similar deductions to (6.113).

Fix $\omega \in \widehat{\mathcal{N}}^c$. For any $n \in \mathbb{N}$ and $i = 1, \dots, \widehat{\ell}_n$, we set $\widehat{t}_i^n(\omega) \triangleq \widehat{t}_i^n \vee \tau(\omega)$ and Proposition 4.1 implies that $(\widehat{\xi}_i^n)^{\tau, \omega} \in \mathcal{F}_{\widehat{t}_i^n(\omega)}^{\tau(\omega)}$ since $\widehat{\xi}_i^n \in \mathcal{F}_{\widehat{t}_i^n}^t \subset \mathcal{F}_{\widehat{t}_i^n(\omega)}^t$. It holds for any $s \in [\tau(\omega), T]$ and $\widetilde{\omega} \in \Omega^{\tau(\omega)}$

$$(\widehat{\Phi}_s^n)^{\tau, \omega}(\widetilde{\omega}) = \widehat{\Phi}_s^n(\omega \otimes_\tau \widetilde{\omega}) = \sum_{i=1}^{\widehat{\ell}_n} \widehat{\xi}_i^n(\omega \otimes_\tau \widetilde{\omega}) \mathbf{1}_{\{s \in (\widehat{t}_i^n, \widehat{t}_{i+1}^n]\}} = \sum_{i=1}^{\widehat{\ell}_n} (\widehat{\xi}_i^n)^{\tau, \omega}(\widetilde{\omega}) \mathbf{1}_{\{s \in (\widehat{t}_i^n(\omega), \widehat{t}_{i+1}^n(\omega)]\}},$$

so $(\widehat{\Phi}_s^n)^{\tau, \omega}$ is an $\mathbb{R}^{k \times d}$ -valued, $\mathbf{F}^{\tau(\omega)}$ -simple process. Applying Proposition 3.2.26 of [27], we see from (6.115) that

$$0 = P_0^{\tau(\omega)} - \lim_{n \rightarrow \infty} \sup_{s \in [\tau(\omega), T]} \left| \int_{\tau(\omega)}^s (\widehat{\Phi}_r^n)^{\tau, \omega} dB_r^{\tau(\omega)} - \int_{\tau(\omega)}^s (\sigma_r^\Theta)^{\tau, \omega} dB_r^{\tau(\omega)} \right|. \quad (6.117)$$

For any $\widetilde{\omega} \in \Omega^{\tau(\omega)}$, one can deduce that

$$\begin{aligned} (\widehat{M}_s^n)^{\tau, \omega}(\widetilde{\omega}) &= \sum_{i=1}^{\widehat{\ell}_n} \widehat{\xi}_i^n(\omega \otimes_\tau \widetilde{\omega}) \left((\omega \otimes_\tau \widetilde{\omega})(s \wedge \widehat{t}_{i+1}^n) - (\omega \otimes_\tau \widetilde{\omega})(s \wedge \widehat{t}_i^n) \right) = \sum_{i=1}^{\widehat{\ell}_n} (\widehat{\xi}_i^n)^{\tau, \omega}(\widetilde{\omega}) \left(\widetilde{\omega}(s \wedge \widehat{t}_{i+1}^n(\omega)) - \widetilde{\omega}(s \wedge \widehat{t}_i^n(\omega)) \right) \\ &= \sum_{i=1}^{\widehat{\ell}_n} (\widehat{\xi}_i^n)^{\tau, \omega}(\widetilde{\omega}) \left(B_{s \wedge \widehat{t}_{i+1}^n(\omega)}^{\tau(\omega)} - B_{s \wedge \widehat{t}_i^n(\omega)}^{\tau(\omega)} \right)(\widetilde{\omega}) = \left(\int_{\tau(\omega)}^s (\widehat{\Phi}_r^n)^{\tau, \omega} dB_r^{\tau(\omega)} \right)(\widetilde{\omega}), \quad s \in [\tau(\omega), T], \end{aligned}$$

which together with (6.116), (6.117) and (6.114) shows that $P_0^{\tau(\omega)}$ -a.s.

$$(M_s^\Theta)^{\tau, \omega} = \int_{\tau(\omega)}^s (\sigma_r^\Theta)^{\tau, \omega} dB_r^{\tau(\omega)} = \int_{\tau(\omega)}^s \sigma(r, (\widetilde{X}^\Theta)_r^{\tau, \omega}, \mu_r^{\tau, \omega}, \nu_r^{\tau, \omega}) dB_r^{\tau(\omega)}, \quad s \in [\tau(\omega), T]. \quad \square \quad (6.118)$$

Proof of Proposition 4.8: Since $(\widetilde{Y}^\Theta, \widetilde{Z}^\Theta, \widetilde{\underline{K}}^\Theta, \widetilde{\overline{K}}^\Theta) \triangleq (\widetilde{Y}^\Theta(T, \xi), \widetilde{Z}^\Theta(T, \xi), \widetilde{\underline{K}}^\Theta(T, \xi), \widetilde{\overline{K}}^\Theta(T, \xi)) \in \mathbb{G}_{\mathbf{F}^t}^q([t, T])$, Corollary 4.3 shows that for P_0^t -a.s. $\omega \in \Omega^t$, the shifted processes $\left\{ \left((\widetilde{Y}^\Theta)_r^{\tau, \omega}, (\widetilde{Z}^\Theta)_r^{\tau, \omega}, (\widetilde{\underline{K}}^\Theta)_r^{\tau, \omega}, (\widetilde{\overline{K}}^\Theta)_r^{\tau, \omega} \right) \right\}_{r \in [\tau(\omega), T]}$

$\in \mathbb{G}_{\mathbf{F}^{\tau(\omega)}}^q([\tau(\omega), T])$. Thus, it suffices to show that for P_0^t -a.s. $\omega \in \Omega^t$,

$$\left((\tilde{Y}^\Theta)^{\tau, \omega}, (Z^\Theta)^{\tau, \omega}, (\underline{K}^\Theta)^{\tau, \omega}, (\tilde{K}^\Theta)^{\tau, \omega} \right) \text{ solves DRBSDE} \left(P_0^{\tau(\omega)}, f_T^{\Theta, \omega}, \underline{L}^{\Theta, \omega}, \bar{L}^{\Theta, \omega} \right). \quad (6.119)$$

As the \mathbf{F}^t -version of $(Y^\Theta(T, \xi), Z^\Theta(T, \xi), \underline{K}^\Theta(T, \xi), \bar{K}^\Theta(T, \xi))$, $(\tilde{Y}^\Theta, \tilde{Z}^\Theta, \tilde{K}^\Theta, \tilde{\bar{K}}^\Theta)$ also satisfies DRBSDE $(P_0^t, f_T^\Theta, \underline{L}^\Theta, \bar{L}^\Theta)$. Thus, it holds except on a P_0^t -null set \mathcal{N} that

$$\begin{cases} \tilde{Y}_{\tau \vee s}^\Theta - \xi + \tilde{K}_T^\Theta - \tilde{K}_{\tau \vee s}^\Theta - \tilde{K}_T^\Theta + \tilde{K}_{\tau \vee s}^\Theta = \int_{\tau \vee s}^T f_T^\Theta(r, \tilde{Y}_r^\Theta, \tilde{Z}_r^\Theta) dr - \int_{\tau \vee s}^T \tilde{Z}_r^\Theta dB_r^t \\ \quad = \int_s^T \mathbf{1}_{\{r > \tau\}} f(r, \tilde{X}_r^\Theta, \tilde{Y}_r^\Theta, \tilde{Z}_r^\Theta, \mu_r, \nu_r) dr - M_T^\Theta + M_s^\Theta, \quad s \in [t, T], \\ \underline{l}(s, \tilde{X}_s^\Theta) \leq \tilde{Y}_s^\Theta \leq \bar{l}(s, \tilde{X}_s^\Theta), \quad s \in [t, T], \text{ and } \int_\tau^T (\tilde{Y}_s^\Theta - \underline{l}(s, \tilde{X}_s^\Theta)) d\tilde{K}_s^\Theta = \int_\tau^T (\bar{l}(s, \tilde{X}_s^\Theta) - \tilde{Y}_s^\Theta) d\tilde{K}_s^\Theta = 0, \end{cases} \quad (6.120)$$

where $M_s^\Theta \triangleq \int_t^s \mathbf{1}_{\{r > \tau\}} \tilde{Z}_r^\Theta dB_r^t$, $s \in [t, T]$.

Fix $\omega \in \Omega^t$. For any $\tilde{\omega} \in (\mathcal{N}^c)^{\tau, \omega}$, applying (6.120) to the path $\omega \otimes_\tau \tilde{\omega}$ over period $[\tau(\omega), T]$ as well as using deductions similar to (6.112) and (6.113), we obtain that

$$\begin{cases} (\tilde{Y}^\Theta)_s^{\tau, \omega}(\tilde{\omega}) - \xi^{\tau, \omega}(\tilde{\omega}) + (\tilde{K}^\Theta)_T^{\tau, \omega}(\tilde{\omega}) - (\tilde{K}^\Theta)_s^{\tau, \omega}(\tilde{\omega}) - (\tilde{K}^\Theta)_T^{\tau, \omega}(\tilde{\omega}) + (\tilde{K}^\Theta)_s^{\tau, \omega}(\tilde{\omega}) + (M^\Theta)_T^{\tau, \omega}(\tilde{\omega}) - (M^\Theta)_s^{\tau, \omega}(\tilde{\omega}) \\ = \int_{\tau(\omega)}^T f(r, (\tilde{X}^\Theta)_r^{\tau, \omega}(\tilde{\omega}), (\tilde{Y}^\Theta)_r^{\tau, \omega}(\tilde{\omega}), (\tilde{Z}^\Theta)_r^{\tau, \omega}(\tilde{\omega}), \mu_r^{\tau, \omega}(\tilde{\omega}), \nu_r^{\tau, \omega}(\tilde{\omega})) dr, \quad s \in [\tau(\omega), T], \\ \underline{l}(s, (\tilde{X}^\Theta)_s^{\tau, \omega}(\tilde{\omega})) \leq (\tilde{Y}_s^\Theta)^{\tau, \omega}(\tilde{\omega}) \leq \bar{l}(s, (\tilde{X}^\Theta)_s^{\tau, \omega}(\tilde{\omega})), \quad s \in [\tau(\omega), T], \text{ and} \\ \int_{\tau(\omega)}^T ((\tilde{Y}_s^\Theta)^{\tau, \omega}(\tilde{\omega}) - \underline{l}(s, (\tilde{X}^\Theta)_s^{\tau, \omega}(\tilde{\omega}))) d(\tilde{K}^\Theta)_s^{\tau, \omega}(\tilde{\omega}) = \int_{\tau(\omega)}^T (\bar{l}(s, (\tilde{X}^\Theta)_s^{\tau, \omega}(\tilde{\omega})) - (\tilde{Y}_s^\Theta)^{\tau, \omega}(\tilde{\omega})) d(\tilde{K}^\Theta)_s^{\tau, \omega}(\tilde{\omega}) = 0. \end{cases}$$

By (4.2) and Corollary 4.2, it holds for P_0^t -a.s. $\omega \in \Omega^t$ that $P_0^{\tau(\omega)}((\mathcal{N}^c)^{\tau, \omega}) = P_0^{\tau(\omega)}((\mathcal{N}^{\tau, \omega})^c) = 1$. Hence, one can deduce from the above system of equations and Proposition 4.7 that for P_0^t -a.s. $\omega \in \Omega^t$, it holds $P_0^{\tau(\omega)}$ -a.s. that

$$\begin{cases} (\tilde{Y}^\Theta)_s^{\tau, \omega} - \xi^{\tau, \omega} + (\tilde{K}^\Theta)_T^{\tau, \omega} - (\tilde{K}^\Theta)_s^{\tau, \omega} - (\tilde{K}^\Theta)_T^{\tau, \omega} + (\tilde{K}^\Theta)_s^{\tau, \omega} + (M^\Theta)_T^{\tau, \omega} - (M^\Theta)_s^{\tau, \omega} \\ = \int_{\tau(\omega)}^T f(r, (\tilde{X}^\Theta)_r^{\tau, \omega}, (\tilde{Y}^\Theta)_r^{\tau, \omega}, (\tilde{Z}^\Theta)_r^{\tau, \omega}, \mu_r^{\tau, \omega}, \nu_r^{\tau, \omega}) dr = \int_{\tau(\omega)}^T f_T^{\Theta, \omega}(r, (\tilde{Y}^\Theta)_r^{\tau, \omega}, (\tilde{Z}^\Theta)_r^{\tau, \omega}) dr, \quad s \in [\tau(\omega), T], \\ \underline{L}_s^{\Theta, \omega} = \underline{l}(s, \tilde{X}_s^{\Theta, \omega}) \leq (\tilde{Y}_s^\Theta)^{\tau, \omega} \leq \bar{l}(s, \tilde{X}_s^{\Theta, \omega}) = \bar{L}_s^{\Theta, \omega}, \quad s \in [\tau(\omega), T], \text{ and} \\ \int_{\tau(\omega)}^T ((\tilde{Y}_s^\Theta)^{\tau, \omega} - \underline{L}_s^{\Theta, \omega}) d(\tilde{K}^\Theta)_s^{\tau, \omega} = \int_{\tau(\omega)}^T (\bar{L}_s^{\Theta, \omega} - (\tilde{Y}_s^\Theta)^{\tau, \omega}) d(\tilde{K}^\Theta)_s^{\tau, \omega} = 0. \end{cases} \quad (6.121)$$

Similar to (6.118), we can deduce from Proposition 3.2.26 and Problem 3.2.27 of [27] (both work for continuous local martingales) that for P_0^t -a.s. $\omega \in \Omega^t$, it holds $P_0^{\tau(\omega)}$ -a.s. that

$$(M^\Theta)_s^{\tau, \omega} = \int_{\tau(\omega)}^s (\mathbf{1}_{\{r > \tau\}} \tilde{Z}_r^\Theta)^{\tau, \omega} dB_r^{\tau(\omega)} = \int_{\tau(\omega)}^s \mathbf{1}_{\{r > \tau(\omega)\}} (\tilde{Z}_r^\Theta)^{\tau, \omega}(\tilde{\omega}) dB_r^{\tau(\omega)} = \int_{\tau(\omega)}^s (\tilde{Z}_r^\Theta)^{\tau, \omega} dB_r^{\tau(\omega)}, \quad s \in [\tau(\omega), T],$$

which together with (6.121) gives (6.119). Therefore, it holds for P_0^t -a.s. $\omega \in \Omega^t$ that

$$\tilde{Y}_s^{\Theta, \omega}(T, \xi^{\tau, \omega})(\tilde{\omega}) = (\tilde{Y}^\Theta(T, \xi))_s^{\tau, \omega}(\tilde{\omega}) = \tilde{Y}_s^\Theta(T, \xi)(\omega \otimes_\tau \tilde{\omega}), \quad \forall (s, \tilde{\omega}) \in [\tau(\omega), T] \times \Omega^{\tau(\omega)}.$$

Taking $(s, \tilde{\omega}) = (\tau(\omega), \Pi_{t, \tau(\omega)}(\omega))$ gives that $\tilde{Y}_{\tau(\omega)}^{\Theta, \omega}(T, \xi^{\tau, \omega}) = (\tilde{Y}_\tau^\Theta(T, \xi))(\omega)$ for P_0^t -a.s. $\omega \in \Omega^t$. \square

Proof of Lemma 4.5: For any $\tilde{\mathcal{E}} \in \mathcal{B}([s, r])$ and $A \in \mathcal{F}_r^s$, applying Lemma 1.2 with $S = T$ yields that

$$\hat{\Pi}_{t, s}^{-1}(\tilde{\mathcal{E}} \times A) = \{(r, \omega) \in [s, T] \times \Omega^t : (r, \Pi_{t, s}(\omega)) \in \tilde{\mathcal{E}} \times A\} = \tilde{\mathcal{E}} \times \Pi_{t, s}^{-1}(A) \in \mathcal{B}([s, r]) \otimes \mathcal{F}_r^t,$$

which shows that all rectangular measurable sets of $\mathcal{B}([s, r]) \otimes \mathcal{F}_r^s$ belongs to $\tilde{\Lambda} \triangleq \{\mathcal{D} \subset [s, r] \times \Omega^s : \hat{\Pi}_{t,s}^{-1}(\mathcal{D}) \in \mathcal{B}([s, r]) \otimes \mathcal{F}_r^s\}$. Clearly, $\tilde{\Lambda}$ is a σ -field of $[s, r] \times \Omega^s$. Thus it follows that $\mathcal{B}([s, r]) \otimes \mathcal{F}_r^s \subset \tilde{\Lambda}$, i.e.,

$$\hat{\Pi}_{t,s}^{-1}(\mathcal{D}) \in \mathcal{B}([s, r]) \otimes \mathcal{F}_r^s, \quad \forall \mathcal{D} \in \mathcal{B}([s, r]) \otimes \mathcal{F}_r^s. \quad (6.122)$$

Next, we show that $(dr \times dP_0^t) \circ \hat{\Pi}_{t,s}^{-1} = (dr \times dP_0^s)$ on $\mathcal{B}([s, T]) \otimes \mathcal{F}_T^s$: For any $\mathcal{E} \in \mathcal{B}([s, T])$ and $A \in \mathcal{F}_T^s$, Lemma 1.2 with $S = T$ again implies that

$$(dr \times dP_0^t)(\hat{\Pi}_{t,s}^{-1}(\mathcal{E} \times A)) = (dr \times dP_0^t)(\mathcal{E} \times \Pi_{t,s}^{-1}(A)) = |\mathcal{E}|P_0^t(\Pi_{t,s}^{-1}(A)) = |\mathcal{E}|P_0^s(A) = (dr \times dP_0^s)(\mathcal{E} \times A),$$

where $|\mathcal{E}|$ denotes the Lebesgue measure of \mathcal{E} . Thus the collection \mathcal{C}_s of all rectangular measurable sets of $\mathcal{B}([s, T]) \otimes \mathcal{F}_T^s$ is contained in $\Lambda \triangleq \{\mathcal{D} \subset [s, T] \times \Omega^s : (dr \times dP_0^t)(\mathcal{D}) = (dr \times dP_0^t)(\hat{\Pi}_{t,s}^{-1}(\mathcal{D}))\}$. In particular, $\emptyset \times \emptyset \in \Lambda$ and $[s, T] \times \Omega^s \in \Lambda$. For any $\mathcal{D} \in \Lambda$, one can deduce that

$$\begin{aligned} (dr \times dP_0^s)(([s, T] \times \Omega^s) \setminus \mathcal{D}) &= (dr \times dP_0^s)([s, T] \times \Omega^s) - (dr \times dP_0^s)(\mathcal{D}) = (dr \times dP_0^t)(\hat{\Pi}_{t,s}^{-1}([s, T] \times \Omega^s)) - (dr \times dP_0^t)(\hat{\Pi}_{t,s}^{-1}(\mathcal{D})) \\ &= (dr \times dP_0^t)(\hat{\Pi}_{t,s}^{-1}([s, T] \times \Omega^s) - \hat{\Pi}_{t,s}^{-1}(\mathcal{D})) = (dr \times dP_0^t)(\hat{\Pi}_{t,s}^{-1}(([s, T] \times \Omega^s) \setminus \mathcal{D})). \end{aligned}$$

On the other hand, for any pairwise-disjoint sequence $\{\mathcal{D}_n\}_{n \in \mathbb{N}}$ of Λ (i.e. $\mathcal{D}_m \cap \mathcal{D}_n = \emptyset$ given $m \neq n$), it is clear that $\{\hat{\Pi}_{t,s}^{-1}(\mathcal{D}_n)\}_{n \in \mathbb{N}}$ is also a pairwise-disjoint sequence. It follows that

$$\begin{aligned} (dr \times dP_0^s)\left(\bigcup_{n \in \mathbb{N}} \mathcal{D}_n\right) &= \sum_{n \in \mathbb{N}} (dr \times dP_0^s)(\mathcal{D}_n) = \sum_{n \in \mathbb{N}} (dr \times dP_0^t)(\hat{\Pi}_{t,s}^{-1}(\mathcal{D}_n)) \\ &= (dr \times dP_0^t)\left(\bigcup_{n \in \mathbb{N}} \hat{\Pi}_{t,s}^{-1}(\mathcal{D}_n)\right) = (dr \times dP_0^t)\left(\hat{\Pi}_{t,s}^{-1}\left(\bigcup_{n \in \mathbb{N}} \mathcal{D}_n\right)\right). \end{aligned}$$

Hence, Λ is a Dynkin system. Since \mathcal{C}_s is closed under intersection, the Dynkin System Theorem shows that $\mathcal{B}([s, T]) \otimes \mathcal{F}_T^s = \sigma(\mathcal{C}_s) \subset \Lambda$, i.e. $(dr \times dP_0^t) \circ \hat{\Pi}_{t,s}^{-1} = (dr \times dP_0^s)$ on $\mathcal{B}([s, T]) \otimes \mathcal{F}_T^s$.

Finally, let us discuss the $\mathcal{P}(\mathbf{F}^t)/\mathcal{P}(\mathbf{F}^s)$ -measurability of $\hat{\Pi}_{t,s}$: Let $\mathcal{D} \in \mathcal{P}(\mathbf{F}^s)$. For any $r \in [s, T]$, since $\mathcal{D} \cap ([s, r] \times \Omega^s) \in \mathcal{B}([s, r]) \otimes \mathcal{F}_r^s$, (6.122) implies that

$$\begin{aligned} (\hat{\Pi}_{t,s}^{-1}(\mathcal{D})) \cap ([t, r] \times \Omega^t) &= (\hat{\Pi}_{t,s}^{-1}(\mathcal{D})) \cap ([s, r] \times \Omega^t) = \hat{\Pi}_{t,s}^{-1}(\mathcal{D}) \cap (\hat{\Pi}_{t,s}^{-1}([s, r] \times \Omega^s)) \\ &= \hat{\Pi}_{t,s}^{-1}(\mathcal{D} \cap ([s, r] \times \Omega^s)) \in \mathcal{B}([s, r]) \otimes \mathcal{F}_r^t \subset \mathcal{B}([t, r]) \otimes \mathcal{F}_r^t. \end{aligned} \quad (6.123)$$

On the other hand, it is clear that $(\hat{\Pi}_{t,s}^{-1}(\mathcal{D})) \cap ([t, r] \times \Omega^t) = \emptyset$ for any $r \in [t, s]$, which together with (6.123) implies that $\hat{\Pi}_{t,s}^{-1}(\mathcal{D}) \in \mathcal{P}(\mathbf{F}^t)$. As $\hat{\Pi}_{t,s}^{-1}(\mathcal{D}) \subset [s, T] \times \Omega^t$, we see that $\hat{\Pi}_{t,s}^{-1}(\mathcal{D}) \in \mathcal{P}_s(\mathbf{F}^t)$. \square

Proof of Proposition 4.9: (1) Let us first discuss the \mathbf{F}^t -progressive measurability of \mathbb{U} -valued process $\hat{\mu}$. Given $s \in [t, T]$ and $U \in \mathcal{B}(\mathbb{U})$, we have to show that $\{(r, \omega) \in [t, s] \times \Omega^t : \hat{\mu}_r(\omega) \in U\} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s^t$: The \mathbf{F}^t -progressive measurability of process μ implies that for any $\mathcal{D} \subset \mathcal{B}([t, s]) \otimes \mathcal{F}_s^t$

$$\{(r, \omega) \in \mathcal{D} : \mu_r(\omega) \in U\} = \{(r, \omega) \in [t, s] \times \Omega^t : \mu_r(\omega) \in U\} \cap \mathcal{D} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s^t, \quad (6.124)$$

which together with (4.8) leads to that

$$\{(r, \omega) \in [t, s] \times \{\tau > s\} : \hat{\mu}_r(\omega) \in U\} = \{(r, \omega) \in [t, s] \times \{\tau > s\} : \mu_r(\omega) \in U\} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s^t.$$

Thus we only need to show that $\{(r, \omega) \in [t, s] \times \{\tau = t_n\} : \hat{\mu}_r(\omega) \in U\} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s^t$ for each $t_n \in [t, s]$:

(i) For $n > N$ with $t_n \leq s$, (4.8) and (6.124) imply that

$$\{(r, \omega) \in [t, s] \times \{\tau = t_n\} : \hat{\mu}_r(\omega) \in U\} = \{(r, \omega) \in [t, s] \times \{\tau = t_n\} : \mu_r(\omega) \in U\} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s^t.$$

(ii) For $n \leq N$ with $t_n \leq s$, let $A_0^n \triangleq \{\tau = t_n\} \setminus \left(\bigcup_{i=1}^n A_i^n\right) \in \mathcal{F}_{t_n}^t$. One can deduce from (4.8) and (6.124) that

$$\{(r, \omega) \in [t, s] \times A_0^n : \hat{\mu}_r(\omega) \in U\} = \{(r, \omega) \in [t, s] \times A_0^n : \mu_r(\omega) \in U\} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s^t. \quad (6.125)$$

For any $i = 1, \dots, \ell_n$, since A_i^n is an $\mathcal{F}_{t_n}^t$ -measurable subset of $\{\tau = t_n\}$, we see from (4.8) that

$$\{(r, \omega) \in [t, s] \times A_i^n : \hat{\mu}_r(\omega) \in U\} = \{(r, \omega) \in [t, t_n] \times A_i^n : \mu_r(\omega) \in U\} \cup \{(r, \omega) \in [t_n, s] \times A_i^n : (\mu_i^n)_r(\Pi_{t, t_n}(\omega)) \in U\}.$$

Clearly, $\{(r, \omega) \in [t, t_n] \times A_i^n : \mu_r(\omega) \in U\} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s^t$ by (6.124). Since $\mathcal{D}_i^n \triangleq \{(r, \tilde{\omega}) \in [t_n, s] \times \Omega^{t_n} : (\mu_i^n)_r(\tilde{\omega}) \in U\} \in \mathcal{B}([t_n, s]) \otimes \mathcal{F}_s^{t_n}$ by the \mathbf{F}^{t_n} -progressive measurability of process μ_i^n , one can deduce from Lemma 4.5 that

$$\begin{aligned} \{(r, \omega) \in [t_n, s] \times A_i^n : (\mu_i^n)_r(\Pi_{t, t_n}(\omega)) \in U\} &= \{(r, \omega) \in [t_n, s] \times A_i^n : (r, \Pi_{t, t_n}(\omega)) \in \mathcal{D}_i^n\} \\ &= \{(r, \omega) \in [t_n, T] \times \Omega^t : \hat{\Pi}_{t, t_n}(r, \omega) \in \mathcal{D}_i^n\} \cap ([t_n, s] \times A_i^n) \\ &= \hat{\Pi}_{t, t_n}^{-1}(\mathcal{D}_i^n) \cap ([t_n, s] \times A_i^n) \in \mathcal{B}([t_n, s]) \otimes \mathcal{F}_s^t \subset \mathcal{B}([t, s]) \otimes \mathcal{F}_s^t. \end{aligned}$$

It follows that $\{(r, \omega) \in [t, s] \times A_i^n : \hat{\mu}_r(\omega) \in U\} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s^t$. Then taking union over $i = 1, \dots, \ell_n$ and combining with (6.125) yield that $\{(r, \omega) \in [t, s] \times \{\tau = t_n\} : \hat{\mu}_r(\omega) \in U\} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s^t$.

(2) For $n = 1, \dots, N$ and $i = 1, \dots, \ell_n$, since

$$\{(r, \omega) \in [t_n, T] \times A_i^n : \hat{\mu}_r(\omega) \in \mathbb{U} \setminus \mathbb{U}_0\} = ([t_n, T] \times A_i^n) \cap \hat{\Pi}_{t, t_n}^{-1}(\{(r, \tilde{\omega}) \in [t_n, T] \times \Omega^{t_n} : (\mu_i^n)_r(\tilde{\omega}) \in \mathbb{U} \setminus \mathbb{U}_0\}),$$

Lemma 4.5 again implies that

$$\begin{aligned} (dr \times dP_0^t)(\{(r, \omega) \in [t_n, T] \times A_i^n : \hat{\mu}_r(\omega) \in \mathbb{U} \setminus \mathbb{U}_0\}) &\leq (dr \times dP_0^t)(\hat{\Pi}_{t, t_n}^{-1}(\{(r, \tilde{\omega}) \in [t_n, T] \times \Omega^{t_n} : (\mu_i^n)_r(\tilde{\omega}) \in \mathbb{U} \setminus \mathbb{U}_0\})) \\ &= (dr \times dP_0^{t_n})(\{(r, \tilde{\omega}) \in [t_n, T] \times \Omega^{t_n} : (\mu_i^n)_r(\tilde{\omega}) \in \mathbb{U} \setminus \mathbb{U}_0\}) = 0. \end{aligned} \quad (6.126)$$

Clearly, $(dr \times dP_0^t)(\{(r, \omega) \in [t, \tau] \times \mathbb{U} : \hat{\mu}_r(\omega) \in \mathbb{U} \setminus \mathbb{U}_0\}) \leq (dr \times dP_0^t)(\{(r, \omega) \in [t, T] \times \Omega^t : \mu_r(\omega) \in \mathbb{U} \setminus \mathbb{U}_0\}) = 0$, which together with (6.126) shows that $\hat{\mu}_r \in \mathbb{U}_0$, $dr \times dP_0^t$ -a.s.

(3) Next, we show that $E_t \int_t^T [\hat{\mu}_r]_{\mathbb{U}}^2 dr < \infty$: By (4.8),

$$\begin{aligned} E_t \int_t^T [\hat{\mu}_r]_{\mathbb{U}}^2 dr &= \int_{\omega \in \Omega^t} \int_t^T [\hat{\mu}_r(\omega)]_{\mathbb{U}}^2 dr dP_0^t(\omega) \\ &= \left(\int_{\omega \in \Omega^t} \int_t^{\tau(\omega)} + \int_{\omega \in A_0} \int_{\tau(\omega)}^T \right) [\mu_r(\omega)]_{\mathbb{U}}^2 dr dP_0^t(\omega) + \sum_{n=1}^N \sum_{i=1}^{\ell_n} \int_{\omega \in A_i^n} \int_{t_n}^T [(\mu_i^n)_r(\Pi_{t, t_n}(\omega))]_{\mathbb{U}}^2 dr dP_0^t(\omega) \\ &\leq \int_{\omega \in \Omega^t} \int_t^T [\mu_r(\omega)]_{\mathbb{U}}^2 dr dP_0^t(\omega) + \sum_{n=1}^N \sum_{i=1}^{\ell_n} \int_{\omega \in \Omega^t} \int_{t_n}^T [(\mu_i^n)_r(\Pi_{t, t_n}(\omega))]_{\mathbb{U}}^2 dr dP_0^t(\omega). \end{aligned}$$

For any $n = 1, \dots, N$ and $i = 1, \dots, \ell_n$, applying Lemma 1.2 with $(s, S) = (t_n, T)$ yields that

$$\begin{aligned} \int_{\omega \in \Omega^t} \int_{t_n}^T [(\mu_i^n)_r(\Pi_{t, t_n}(\omega))]_{\mathbb{U}}^2 dr dP_0^t(\omega) &= \int_{\tilde{\omega} \in \Omega^{t_n}} \int_{t_n}^T [(\mu_i^n)_r(\tilde{\omega})]_{\mathbb{U}}^2 dr dP_0^t(\Pi_{t, t_n}^{-1}(\tilde{\omega})) \\ &= \int_{\tilde{\omega} \in \Omega^{t_n}} \int_{t_n}^T [(\mu_i^n)_r(\tilde{\omega})]_{\mathbb{U}}^2 dr dP_0^{t_n}(\tilde{\omega}) + E_{t_n} \int_{t_n}^T [(\mu_i^n)_r]_{\mathbb{U}}^2 dr < \infty. \end{aligned}$$

Thus it follows that

$$E_t \int_t^T [\hat{\mu}_r]_{\mathbb{U}}^2 dr \leq E_t \int_t^T [\mu_r]_{\mathbb{U}}^2 dr + \sum_{n=1}^N \sum_{i=1}^{\ell_n} E_{t_n} \int_{t_n}^T [(\mu_i^n)_r]_{\mathbb{U}}^2 dr < \infty, \quad (6.127)$$

which together with part (1) shows that $\hat{\mu} \in \mathcal{U}^t$.

(4) Let $(r, \omega) \in [\tau, T]_{A_i^n} = [t_n, T] \times A_i^n$ for some $n = 1, \dots, N$ and $i = 1, \dots, \ell_n$. For any $\tilde{\omega} \in \Omega^{t_n}$, since $\omega \otimes_{t_n} \tilde{\omega} \in A_i^n$ by Lemma 4.1, it follows from (4.8) that $\hat{\mu}_r^{t_n, \omega}(\tilde{\omega}) = \hat{\mu}_r(\omega \otimes_{t_n} \tilde{\omega}) = (\mu_i^n)_r(\Pi_{t, t_n}(\omega \otimes_{t_n} \tilde{\omega})) = (\mu_i^n)_r(\tilde{\omega})$.

On the other hand, we consider $(r, \omega) \in [\tau, T]_{A_0}$. For any $\tilde{\omega} \in \Omega^{\tau(\omega)}$, we claim that

$$\omega \otimes_{\tau} \tilde{\omega} \in A_0. \quad (6.128)$$

Assume not, i.e. $\omega \otimes_\tau \tilde{\omega} \in A_i^n$ for some $n = 1, \dots, N$ and $i = 1, \dots, \ell_n$. By Lemma 4.3, $\tau(\omega) = \tau(\omega \otimes_\tau \tilde{\omega}) = t_n$. So $\omega \otimes_{t_n} \tilde{\omega} = \omega \otimes_\tau \tilde{\omega} \in A_i^n$ and Lemma 4.1 shows that $\omega \in A_i^n$, a contradiction appears. Thus $\omega \otimes_\tau \tilde{\omega} \in A_0$. As $r \geq \tau(\omega) = \tau(\omega \otimes_\tau \tilde{\omega})$, we see that $(r, \omega \otimes_\tau \tilde{\omega}) \in [\tau, T]_{A_0}$. Then (4.8) yields that $\hat{\mu}_r^{\tau, \omega}(\tilde{\omega}) = \hat{\mu}_r(\omega \otimes_\tau \tilde{\omega}) = \mu_r(\omega \otimes_\tau \tilde{\omega}) = \mu_r^{\tau, \omega}(\tilde{\omega})$. \square

Proof of Proposition 4.10: (1) We first assume that $\alpha \in \mathcal{A}^t$ and $\{\alpha_i^n\}_{i=1}^{\ell_n} \subset \mathcal{A}^{t_n}$ for $n = 1, \dots, N$. Then (2.25) holds for all $(r, \omega) \in [t, T] \times \Omega^t$ except on a $dr \times dP_0^t$ -null set \mathcal{D} . And for $n = 1, \dots, N$ and $i = 1, \dots, \ell_n$, α_i^n satisfy (2.25) for some $\kappa_i^n > 0$ and some non-negative \mathbf{F}^{t_n} -measurable process $\Psi^{n,i}$ with $E_{t_n} \int_{t_n}^T (\Psi_s^{n,i})^2 ds < \infty$: i.e., it holds all $(r, \tilde{\omega}) \in [t_n, T] \times \Omega^{t_n}$ except on a $dr \times dP_0^{t_n}$ -null set \mathcal{D}_i^n that

$$[\alpha_i^n(r, \tilde{\omega}, v)]_{\mathbb{U}} \leq \Psi_r^{n,i}(\tilde{\omega}) + \kappa_i^n[v]_{\mathbb{V}}, \quad \forall v \in \mathbb{V}.$$

Fix $n = 1, \dots, N$ and $i = 1, \dots, \ell_n$. The $\mathcal{P}_{t_n}(\mathbf{F}^t)/\mathcal{P}(\mathbf{F}^{t_n})$ -measurability of the mapping $\hat{\Pi}_{t,t_n} : [t_n, T] \times \Omega^t \rightarrow [t_n, T] \times \Omega^{t_n}$ by Lemma 4.5 shows that the function $\alpha_i^n(\hat{\Pi}_{t,t_n}(r, \omega), v) = \alpha_i^n(r, \Pi_{t,t_n}(\omega), v)$, $\forall (r, \omega, v) \in [t_n, T] \times \Omega^t \times \mathbb{V}$ is $\mathcal{P}_{t_n}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{V})/\mathcal{B}(\mathbb{U})$ -measurable, which together with the fact $[\tau, T]_{A_i^n} = [t_n, T] \times A_i^n \in \mathcal{P}(\mathbf{F}^t)$ implies that the function $\hat{\alpha}$ is $\mathcal{P}(\mathbf{F}^t) \otimes \mathcal{B}(\mathbb{V})/\mathcal{B}(\mathbb{U})$ -measurable.

Similar to (6.126), one can deduce from Lemma 4.5 that

$$(dr \times dP_0^t)(\{(r, \omega) \in [t_n, T] \times A_i^n : \hat{\alpha}(r, \omega, \mathbb{V}_0) \setminus \mathbb{U}_0 \neq \emptyset\}) \leq (dr \times dP_0^{t_n})(\{(r, \tilde{\omega}) \in [t_n, T] \times \Omega^{t_n} : \alpha_i^n(r, \tilde{\omega}, \mathbb{V}_0) \setminus \mathbb{U}_0 \neq \emptyset\}) = 0.$$

As $(dr \times dP_0^t)(\{(r, \omega) \in [t, \tau] \cup [\tau, T]_{A_0} : \hat{\alpha}(r, \omega, \mathbb{V}_0) \setminus \mathbb{U}_0 \neq \emptyset\}) \leq (dr \times dP_0^t)(\{(r, \omega) \in [t, T] \times \Omega^t : \alpha(r, \omega, \mathbb{V}_0) \setminus \mathbb{U}_0 \neq \emptyset\}) = 0$, we see that $\hat{\alpha}(r, \mathbb{V}_0) \subset \mathbb{U}_0$, $dr \times dP_0^t$ -a.s.

Since the mapping $\hat{\Pi}_{t,t_n}$ is also $\mathcal{B}([t_n, T]) \otimes \mathcal{F}_T^t/\mathcal{B}([t_n, T]) \otimes \mathcal{F}_T^{t_n}$ -measurable by Lemma 4.5 again, the process $\Psi^{n,i}(\hat{\Pi}_{t,t_n}(r, \omega)) = \Psi^{n,i}(r, \Pi_{t,t_n}(\omega))$, $\forall (r, \omega) \in [t_n, T] \times \Omega^t$ is $\mathcal{B}([t_n, T]) \otimes \mathcal{F}_T^t/\mathcal{B}(\mathbb{R})$ -measurable, which together with the fact $[\tau, T]_{A_i^n} = [t_n, T] \times A_i^n \in \mathcal{B}([t, T]) \otimes \mathcal{F}_T^t$ gives rise to a non-negative measurable process on $(\Omega^t, \mathcal{F}_T^t)$:

$$\hat{\Psi}_r(\omega) \triangleq \begin{cases} \Psi_r^{n,i}(\Pi_{t,t_n}(\omega)), & \text{if } (r, \omega) \in [\tau, T]_{A_i^n} = [t_n, T] \times A_i^n \text{ for } n = 1, \dots, N \text{ and } i = 1, \dots, \ell_n, \\ \Psi_r(\omega), & \text{if } (r, \omega) \in [t, \tau] \cup [\tau, T]_{A_0}. \end{cases}$$

Similar to (6.127), one can show that $E_t \int_t^T \hat{\Psi}_r^2 dr \leq E_t \int_t^T \Psi_r^2 dr + \sum_{n=1}^N \sum_{i=1}^{\ell_n} E_{t_n} \int_{t_n}^T (\Psi_r^{n,i})^2 dr < \infty$.

Let $\hat{\kappa} \triangleq \kappa \vee \max\{\kappa_i^n : n = 1, \dots, N \text{ and } i = 1, \dots, \ell_n\} > 0$. For any $(r, \omega) \in ([t, \tau] \cup [\tau, T]_{A_0}) \setminus \mathcal{D}$, one has

$$[\hat{\alpha}(r, \omega, v)]_{\mathbb{U}} = [\alpha(r, \omega, v)]_{\mathbb{U}} \leq \Psi_r(\omega) + \hat{\kappa}[v]_{\mathbb{V}} = \hat{\Psi}_r(\omega) + \hat{\kappa}[v]_{\mathbb{V}}, \quad \forall v \in \mathbb{V};$$

On the other hand, if $(r, \omega) \in [t_n, T] \times A_i^n \setminus \hat{\Pi}_{t,t_n}^{-1}(\mathcal{D}_i^n)$ for some $n = 1, \dots, N$ and $i = 1, \dots, \ell_n$, then $(r, \Pi_{t,t_n}(\omega)) = \hat{\Pi}_{t,t_n}(r, \omega) \in ([t_n, T] \times \Omega^{t_n}) \setminus \mathcal{D}_i^n$ and it follows that

$$[\hat{\alpha}(r, \omega, v)]_{\mathbb{U}} = [\alpha_i^n(r, \Pi_{t,t_n}(\omega), v)]_{\mathbb{U}} \leq \Psi_r^{n,i}(\Pi_{t,t_n}(\omega)) + \hat{\kappa}[v]_{\mathbb{V}} = \hat{\Psi}_r(\omega) + \hat{\kappa}[v]_{\mathbb{V}}, \quad \forall v \in \mathbb{V}.$$

Since $\tilde{\mathcal{D}} \triangleq \mathcal{D} \cup \left(\bigcup_{n=1}^N \bigcup_{i=1}^{\ell_n} \hat{\Pi}_{t,t_n}^{-1}(\mathcal{D}_i^n) \right)$ is a $dr \times dP_0^t$ -null set by Lemma 4.5, we see that $\hat{\alpha}$ satisfies (2.25) $dr \times dP_0^t$ -a.s. Therefore, $\hat{\alpha}$ is an \mathcal{A}^t -strategy.

(2) Next, let us verify (4.10): Fix $\nu \in \mathcal{V}^t$. For $n = 1, \dots, N$, Proposition 4.6 (1) shows that $\nu^{t_n, \omega} \in \mathcal{V}^{t_n}$ for all $\omega \in \Omega^t$ except on a P_0^t -null set \mathcal{N}_n . Let $\omega \in \bigcap_{n=1}^N \mathcal{N}_n^c$ and $r \in [\tau(\omega), T]$. If $\omega \in A_i^n$ for some $n = 1, \dots, N$ and $i = 1, \dots, \ell_n$, then $r \geq \tau(\omega) = t_n$. For any $\tilde{\omega} \in \Omega^{t_n}$, since $\omega \otimes_{t_n} \tilde{\omega} \in A_i^n$ by Lemma 4.1, it follows from (4.9) that

$$(\hat{\alpha}\langle \nu \rangle)_r^{t_n, \omega}(\tilde{\omega}) = (\hat{\alpha}\langle \nu \rangle)_r(\omega \otimes_{t_n} \tilde{\omega}) = \hat{\alpha}(r, \omega \otimes_{t_n} \tilde{\omega}, \nu_r(\omega \otimes_{t_n} \tilde{\omega})) = \alpha_i^n(r, \tilde{\omega}, \nu_r^{t_n, \omega}(\tilde{\omega})) = (\alpha_i^n\langle \nu^{t_n, \omega} \rangle)_r(\tilde{\omega}).$$

Otherwise, if $\omega \in A_0$, we have seen from (6.128) that $\omega \otimes_\tau \Omega^{\tau(\omega)} \subset A_0$. For any $\tilde{\omega} \in \Omega^{\tau(\omega)}$, since $r \geq \tau(\omega) = \tau(\omega \otimes_\tau \tilde{\omega})$ by Lemma 4.3, we see that $(r, \omega \otimes_\tau \tilde{\omega}) \in [\tau, T]_{A_0}$. Then (6.128) leads to that

$$(\hat{\alpha}\langle \nu \rangle)_r^{\tau, \omega}(\tilde{\omega}) = (\hat{\alpha}\langle \nu \rangle)_r(\omega \otimes_\tau \tilde{\omega}) = \hat{\alpha}(r, \omega \otimes_\tau \tilde{\omega}, \nu_r(\omega \otimes_\tau \tilde{\omega})) = \alpha(r, \omega \otimes_\tau \tilde{\omega}, \nu_r(\omega \otimes_\tau \tilde{\omega})) = (\alpha\langle \nu \rangle)_r^{\tau, \omega}(\tilde{\omega}).$$

(3) Now, let $\alpha \in \hat{\mathcal{A}}^t$ and $\{\alpha_i^n\}_{i=1}^{\ell_n} \subset \hat{\mathcal{A}}^{t_n}$ for $n = 1, \dots, N$. We shall show that $\hat{\alpha}$ satisfies (2.26), and it thus belongs to $\hat{\mathcal{A}}^t$. Fix $\varepsilon > 0$. There exist $\delta > 0$ and a closed subset F of Ω^t with $P_0^t(F) > 1 - \frac{\varepsilon}{2}$ such that for any $\omega, \omega' \in F$ with $\|\omega - \omega'\|_t < \delta$

$$\sup_{r \in [t, T]} \sup_{v \in \mathbb{V}} \rho_{\mathbb{U}}(\alpha(r, \omega, v), \alpha(r, \omega', v)) < \frac{\varepsilon}{2}. \quad (6.129)$$

As $A_0 \in \mathcal{F}_T^t = \mathcal{B}(\Omega^t)$ by (1.4), we can find a closed subset F_0 of Ω^t that is included in A_0 and satisfies $P_0^t(A_0 \setminus F_0) < \frac{\varepsilon}{8}$ (see e.g. Proposition 15.11 of [40]).

Let $\ell_* \triangleq \sum_{n=1}^N \ell_n$. Given $n = 1, \dots, N$ and $i = 1, \dots, \ell_n$, there exist $\delta_i^n > 0$ and a closed subset \tilde{F}_i^n of Ω^{t_n} with $P_0^{t_n}(\tilde{F}_i^n) > 1 - \frac{\varepsilon}{4\ell_*}$ such that for any $\tilde{\omega}, \tilde{\omega}' \in \tilde{F}_i^n$ with $\|\tilde{\omega} - \tilde{\omega}'\|_{t_n} < \delta_i^n$

$$\sup_{r \in [t_n, T]} \sup_{v \in \mathbb{V}} \rho_{\mathbb{U}}(\alpha_i^n(r, \tilde{\omega}, v), \alpha_i^n(r, \tilde{\omega}', v)) < \frac{\varepsilon}{4\ell_*}. \quad (6.130)$$

Applying Lemma 1.2 with $(s, S) = (t_n, T)$ shows that $\Pi_{t, t_n}^{-1}(\tilde{F}_i^n)$ is a closed subset of Ω^t and that

$$P_0^t(\Pi_{t, t_n}^{-1}(\tilde{F}_i^n)) = P_0^{t_n}(\tilde{F}_i^n) > 1 - \frac{\varepsilon}{4\ell_*}. \quad (6.131)$$

Similar to F_0 , one can find a closed subset F_i^n of Ω^t that is included in A_i^n and satisfies $P_0^t(A_i^n \setminus F_i^n) < \frac{\varepsilon}{8\ell_*}$. Then

$$P_0^t(F_0) + \sum_{n=1}^N \sum_{i=1}^{\ell_n} P_0^t(F_i^n) = P_0^t(A_0) - P_0^t(A_0 \setminus F_0) + \sum_{n=1}^N \sum_{i=1}^{\ell_n} P_0^t(A_i^n) - \sum_{n=1}^N \sum_{i=1}^{\ell_n} P_0^t(A_i^n \setminus F_i^n) > 1 - \frac{\varepsilon}{4}. \quad (6.132)$$

Since $\hat{F}_0 \triangleq F \cap F_0$ and $\hat{F}_i^n \triangleq F \cap F_i^n \cap \Pi_{t, t_n}^{-1}(\tilde{F}_i^n)$, $n = 1, \dots, N$ and $i = 1, \dots, \ell_n$ are disjoint closed subsets of Ω^t , we let $\underline{\delta} > 0$ stand for the minimal distance between any two of them. Let

$$\hat{F} \triangleq \hat{F}_0 \cup \left(\bigcup \{ \hat{F}_i^n : n = 1, \dots, N \text{ and } i = 1, \dots, \ell_n \} \right) \text{ and } \hat{\delta} \triangleq \left(\frac{1}{2} \underline{\delta} \right) \wedge \delta \wedge \frac{1}{2} \min \{ \delta_i^n, n = 1, \dots, N \text{ and } i = 1, \dots, \ell_n \}.$$

For any $A_1, A_2 \in \mathcal{F}_T^t$, one has

$$P_0^t(A_1 \cap A_2) = P_0^t(A_1) + P_0^t(A_2) - P_0^t(A_1 \cup A_2) \geq P_0^t(A_1) + P_0^t(A_2) - 1. \quad (6.133)$$

Taking $A_1 = F$ and $A_2 = \left(\bigcup_{n=1}^N \bigcup_{i=1}^{\ell_n} F_i^n \right) \cup F_0$, we can deduce from (6.132) that

$$P_0^t(F \cap F_0) + \sum_{n=1}^N \sum_{i=1}^{\ell_n} P_0^t(F \cap F_i^n) > 1 - \frac{3}{4} \varepsilon. \quad (6.134)$$

Also, for $n = 1, \dots, N$ and $i = 1, \dots, \ell_n$, letting $A_1 = F \cap F_i^n$ and $A_2 = \Pi_{t, t_n}^{-1}(\tilde{F}_i^n)$ in (6.133), we see from (6.131) that

$$P_0^t(\hat{F}_i^n) = P_0^t\left(F \cap F_i^n \cap \Pi_{t, t_n}^{-1}(\tilde{F}_i^n)\right) > P_0^t(F \cap F_i^n) - \frac{\varepsilon}{4\ell_*},$$

which together with (6.134) leads to that

$$P_0^t(\hat{F}) = P_0^t(\hat{F}_0) + \sum_{n=1}^N \sum_{i=1}^{\ell_n} P_0^t(\hat{F}_i^n) > 1 - \varepsilon.$$

Now let $\omega, \omega' \in \hat{F}$ with $\|\omega - \omega'\|_t < \hat{\delta}$. If $\omega \in \hat{F}_0$, so is ω' since $\hat{\delta} \leq \frac{1}{2} \underline{\delta}$. As $\hat{F}_0 \subset F \cap A_0$ and $\hat{\delta} \leq \delta$, it follows from (4.9) and (6.129) that

$$\sup_{r \in [t, T]} \sup_{v \in \mathbb{V}} \rho_{\mathbb{U}}(\hat{\alpha}(r, \omega, v), \hat{\alpha}(r, \omega', v)) = \sup_{r \in [t, T]} \sup_{v \in \mathbb{V}} \rho_{\mathbb{U}}(\alpha(r, \omega, v), \alpha(r, \omega', v)) < \frac{\varepsilon}{2}.$$

On the other hand, if $\omega \in \widehat{F}_i^n$ for some $n = 1, \dots, N$ and $i = 1, \dots, \ell_n$, so is ω' . Since $\widehat{F}_i^n \subset F \cap A_i^n \cap \Pi_{t, t_n}^{-1}(\widehat{F}_i^n)$ and

$$\sup_{r \in [t_n, T]} |(\Pi_{t, t_n}(\omega'))(r) - (\Pi_{t, t_n}(\omega))(r)| \leq |\omega'(t_n) - \omega(t_n)| + \sup_{r \in [t_n, T]} |\omega'(r) - \omega(r)| \leq 2 \sup_{r \in [t, T]} |\omega'(r) - \omega(r)| < 2\widehat{\delta} \leq \delta_i^n,$$

we can deduce from (4.9) and (6.130) that

$$\begin{aligned} \sup_{r \in [t, T]} \sup_{v \in \mathbb{V}} \rho_{\mathbb{U}}(\widehat{\alpha}(r, \omega, v), \widehat{\alpha}(r, \omega', v)) &\leq \sup_{r \in [t, t_n]} \sup_{v \in \mathbb{V}} \rho_{\mathbb{U}}(\alpha(r, \omega, v), \alpha(r, \omega', v)) \\ &+ \sup_{r \in [t_n, T]} \sup_{v \in \mathbb{V}} \rho_{\mathbb{U}}(\alpha_i^n(r, \Pi_{t, t_n}(\omega), v), \alpha_i^n(r, \Pi_{t, t_n}(\omega'), v)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4\ell_*} < \varepsilon. \end{aligned}$$

Therefore, $\widehat{\alpha}$ satisfies (2.26), to wit, $\widehat{\alpha} \in \widehat{\mathcal{A}}^t$. \square

6.5 Proofs of Section 5

Proof of Lemma 5.1: As a subspace of $\mathbb{R}^{d \times d}$, \mathbb{S}_d is also a normed vector space (We can regard the restriction to \mathbb{S}_d of the Euclidean norm $|\cdot|$ on $\mathbb{R}^{d \times d}$ as the Euclidean norm on \mathbb{S}_d .) Since each $\mathbb{R}^{d \times d}$ -valued symmetric matrix is uniquely determined by its lower (or upper) triangle, we see that

$$\phi(\Gamma) \triangleq (g_{11}, g_{21}, g_{22}, g_{31}, g_{32}, g_{33}, \dots, g_{d1}, \dots, g_{dd}), \quad \forall \Gamma = (g_{ij})_{i,j=1}^d \in \mathbb{S}_d$$

defines a bijection between \mathbb{S}_d and $\mathbb{R}^{\frac{d(1+d)}{2}}$. Clearly, $|\phi(\Gamma)| \leq |\Gamma| \leq \sqrt{2}|\phi(\Gamma)|$, $\forall \Gamma \in \mathbb{S}_d$, thus ϕ is a homeomorphism. Then the separability of $\mathbb{R}^{\frac{d(1+d)}{2}}$ leads to that of \mathbb{S}_d .

Moreover, as $\det(\cdot)$ is a continuous function on $\mathbb{R}^{d \times d}$, its restriction on \mathbb{S}_d is also continuous w.r.t. the relative Euclidean topology on \mathbb{S}_d . \square

Proof of Lemma 5.2: Let $i, j \in \{1, \dots, d\}$. For any $n \in \mathbb{N}$, we set $\tau_0^{n,i} = t$ and recursively define \mathbf{F}^t -stopping times

$$\tau_\ell^{n,i} \triangleq \inf \left\{ s \in [\tau_{\ell-1}^{n,i}, T] : \left| B_s^i - B_{\tau_{\ell-1}^{n,i}}^i \right| > 2^{-n} \right\} \wedge T, \quad \forall \ell \in \mathbb{N}.$$

Clearly, $\mathcal{J}_s^{n,i,j} \triangleq \lim_{m \rightarrow \infty} \sum_{\ell=1}^m B_{\tau_{\ell-1}^{n,i} \wedge s}^{t,i} (B_{\tau_\ell^{n,i} \wedge s}^{t,j} - B_{\tau_{\ell-1}^{n,i} \wedge s}^{t,j})$, $s \in [t, T]$ is an $\mathbb{R} \cup \{-\infty\}$ -valued, \mathbf{F}^t -progressively measurable process, so is $\mathcal{J}^{i,j} \triangleq \lim_{n \rightarrow \infty} \mathcal{J}^{n,i,j}$.

For any $P \in \mathcal{Q}^t$, as Lemma 1.3 (1) shows that B^t is also a continuous semi-martingale with respect to (\mathbf{F}^P, P) , we know from Theorem 2 of [26] that $\lim_{n \rightarrow \infty} \sup_{s \in [t, T]} \left| \mathcal{J}_s^{n,i,j} - \int_{[t,s]}^P B_r^{t,i} dB_r^{t,j} \right| = 0$, P -a.s. Thus it holds P -a.s. that

$$\mathcal{J}_s^{i,j} = \int_{[t,s]}^P B_r^{t,i} dB_r^{t,j}, \quad \forall s \in [t, T]. \quad (6.135)$$

This gives rise to a pathwise definition of the (i, j) -th cross variance of B^t as well as its density:

$$\langle B^{t,i}, B^{t,j} \rangle_s \triangleq B^{t,i} B^{t,j} - \mathcal{J}_s^{i,j} - \mathcal{J}_s^{j,i} \quad \text{and} \quad \hat{a}_s^{t,i,j} \triangleq \lim_{m \rightarrow \infty} m \left(\langle B^{t,i}, B^{t,j} \rangle_s - \langle B^{t,i}, B^{t,j} \rangle_{(s-1/m)^+} \right), \quad s \in [t, T],$$

both of which are $\mathbb{R} \cup \{\infty\}$ -valued, \mathbf{F}^t -progressively measurable processes.

For any $P \in \mathcal{Q}^t$, we see from (6.135) that P -a.s.

$$\langle B^{t,i}, B^{t,j} \rangle_s = B^{t,i} B^{t,j} - \int_{[t,s]}^P B_r^{t,i} dB_r^{t,j} - \int_{[t,s]}^P B_r^{t,j} dB_r^{t,i} = \langle B^{t,i}, B^{t,j} \rangle_s^P, \quad \forall s \in [t, T].$$

Then (5.4) easily follows. \square

Proof of Lemma 5.3: Let $t \in [0, T]$. We see from Lemma 5.2 that $|\hat{a}^t|$ is a $[0, \infty]$ -valued, \mathbf{F}^t -progressively measurable process. It follows that $\mathbf{1}_{\{|\hat{a}^t| \in (0, \infty)\}} \frac{1}{|\hat{a}^t|}$ is a $[0, \infty)$ -valued, \mathbf{F}^t -progressively measurable process and thus that $\mathbf{n}^t \triangleq \mathbf{1}_{\{|\hat{a}^t| \in (0, \infty)\}} \frac{\hat{a}^t}{|\hat{a}^t|}$ is an \mathbb{S}_d -valued, \mathbf{F}^t -progressively measurable process. Since the determinant $\det(\cdot)$ is continuous on \mathbb{S}_d by Lemma 5.1, $\hat{\mathbf{n}}^t \triangleq \mathbf{1}_{\{\det(\mathbf{n}^t) > 0\}} \mathbf{n}^t + \mathbf{1}_{\{\det(\mathbf{n}^t) \leq 0\}} I_{d \times d}$ defines an $\mathbb{S}_d^{>0}$ -valued, \mathbf{F}^t -progressively measurable process.

For any $j \in \mathbb{N}$, let $c_j \triangleq -\frac{1 \times 3 \times \cdots \times (2j-3)}{2^j j!}$, which is the j -th coefficient of the power series of $\sqrt{1-x}$, $x \in [-1, 1]$. When a $\Gamma \in \mathbb{S}_d^{>0}$ has $|\Gamma| \leq 1$, we know (see e.g. Theorem VI.9 of [38]) that $\varsigma \triangleq I_{d \times d} + \sum_{j \in \mathbb{N}} c_j (I_{d \times d} - \Gamma)^j$ is the unique element in $\mathbb{S}_d^{>0}$ such that $\varsigma^2 = \varsigma \cdot \varsigma = \Gamma$. Consequently, $q^t \triangleq I_{d \times d} + \sum_{j \in \mathbb{N}} c_j (I_{d \times d} - \hat{\mathbf{n}}^t)^j$ is the unique $\mathbb{S}_d^{>0}$ -valued, \mathbf{F}^t -progressively measurable process such that

$$(q^t)^2 = \hat{\mathbf{n}}^t = \mathbf{1}_{\{|\hat{a}^t| \in (0, \infty)\}} \mathbf{1}_{\{\det(\hat{a}^t) > 0\}} \frac{\hat{a}^t}{|\hat{a}^t|} + (\mathbf{1}_{\{|\hat{a}^t| \in (0, \infty)\}} \mathbf{1}_{\{\det(\hat{a}^t) \leq 0\}} + \mathbf{1}_{\{|\hat{a}^t| = 0 \text{ or } \infty\}}) I_{d \times d}.$$

It follows that $\hat{q}^t \triangleq q^t (\mathbf{1}_{\{|\hat{a}^t| \in (0, \infty)\}} \sqrt{|\hat{a}^t|} + \mathbf{1}_{\{|\hat{a}^t| = 0 \text{ or } \infty\}})$ is the unique $\mathbb{S}_d^{>0}$ -valued, \mathbf{F}^t -progressively measurable process satisfying

$$(\hat{q}^t)^2 = \mathbf{1}_{\{|\hat{a}^t| \in (0, \infty)\}} \mathbf{1}_{\{\det(\hat{a}^t) > 0\}} \hat{a}^t + \mathbf{1}_{\{|\hat{a}^t| \in (0, \infty)\}} \mathbf{1}_{\{\det(\hat{a}^t) \leq 0\}} |\hat{a}^t| I_{d \times d} + \mathbf{1}_{\{|\hat{a}^t| = 0 \text{ or } \infty\}} I_{d \times d}. \quad (6.136)$$

Given $P \in \mathcal{Q}_W^t$, since $|\Gamma| \in (0, \infty)$ for each $\Gamma \in \mathbb{S}_d^{>0}$, we can deduce from the second part of (5.5) and (6.136) that P -a.s., $(\hat{q}^t)_s^2 = \hat{q}_s^t \cdot \hat{q}_s^t = \hat{a}_s^t$ for a.e. $s \in [t, T]$. \square

Proof of Lemma 5.4: Let $s \in [t, T]$. For any $r \in [t, s]$ and $\mathcal{E} \in \mathcal{B}(\mathbb{R}^d)$, similar to (5.10), one can deduce from the \mathbf{F}^t -adapttness of $\tilde{X}^{t,x,\mu}$ that

$$(\mathcal{X}^{t,x,\mu})^{-1} \left((B_r^t)^{-1}(\mathcal{E}) \right) = \left\{ \omega \in \Omega^t : \mathcal{X}_r^{t,x,\mu}(\omega) \in \mathcal{E} \right\} = \begin{cases} \mathcal{N}_\mu^{t,x} \cup \left((\mathcal{N}_\mu^{t,x})^c \cap \{ \omega \in \Omega^t : \tilde{X}_r^{t,x,\mu}(\omega) \in \mathcal{E} \} \right) \in \overline{\mathcal{F}}_s^t, & \text{if } 0 \in \mathcal{E}, \\ (\mathcal{N}_\mu^{t,x})^c \cap \{ \omega \in \Omega^t : \tilde{X}_r^{t,x,\mu}(\omega) \in \mathcal{E} \} \in \overline{\mathcal{F}}_s^t, & \text{if } 0 \notin \mathcal{E}. \end{cases}$$

where $\mathcal{E}_x = \{x + x' : x' \in \mathcal{E}\} \in \mathcal{B}(\mathbb{R}^d)$. Thus $(B_r^t)^{-1}(\mathcal{E}) \in \Lambda_s^t \triangleq \{A \subset \Omega^t : (\mathcal{X}^{t,x,\mu})^{-1}(A) \in \overline{\mathcal{F}}_s^t\}$. Clearly, Λ_s^t is a σ -field of Ω^t . So it follows that

$$\mathcal{F}_s^t = \sigma \left((B_r^t)^{-1}(\mathcal{E}); r \in [t, s], \mathcal{E} \in \mathcal{B}(\mathbb{R}^d) \right) \subset \Lambda_s^t. \quad (6.137)$$

For any $\mathcal{N} \in \mathcal{N}^{P^{t,x,\mu}}$, it is contained in some $A \in \mathcal{F}_T^t$ with $P^{t,x,\mu}(A) = 0$. By (5.11) and (5.12), $(\mathcal{X}^{t,x,\mu})^{-1}(A) \in \mathcal{F}_T^t$ and $P_0^t \left((\mathcal{X}^{t,x,\mu})^{-1}(A) \right) = P^{t,x,\mu}(A) = 0$. Then, as a subset of $(\mathcal{X}^{t,x,\mu})^{-1}(A)$,

$$(\mathcal{X}^{t,x,\mu})^{-1}(\mathcal{N}) \in \mathcal{N}^{P_0^t} \subset \overline{\mathcal{F}}_s^t. \quad (6.138)$$

Thus, $\mathcal{N}^{P^{t,x,\mu}} \subset \Lambda_s^t$, which together with (6.137) yields $\mathcal{F}_s^{P^{t,x,\mu}} = \sigma(\mathcal{F}_s^t \cup \mathcal{N}^{P^{t,x,\mu}}) \subset \Lambda_s^t$, i.e. $(\mathcal{X}^{t,x,\mu})^{-1}(\mathcal{F}_s^{P^{t,x,\mu}}) \subset \overline{\mathcal{F}}_s^t$.

For any $A \in \mathcal{F}_T^{P^{t,x,\mu}}$, we know (see e.g. Proposition 11.4 of [40]) that $A = \tilde{A} \cup \mathcal{N}$ for some $\tilde{A} \in \mathcal{F}_T^t$ and $\mathcal{N} \in \mathcal{N}^{P^{t,x,\mu}}$. Since $(\mathcal{X}^{t,x,\mu})^{-1}(\tilde{A}) \in \mathcal{F}_T^t$ by (5.11) and since $(\mathcal{X}^{t,x,\mu})^{-1}(\mathcal{N}) \in \mathcal{N}^{P_0^t}$ by (6.138), one can deduce that

$$P_0^t \circ (\mathcal{X}^{t,x,\mu})^{-1}(A) = P_0^t \left((\mathcal{X}^{t,x,\mu})^{-1}(\tilde{A}) \cup (\mathcal{X}^{t,x,\mu})^{-1}(\mathcal{N}) \right) = P_0^t \left((\mathcal{X}^{t,x,\mu})^{-1}(\tilde{A}) \right) = P^{t,x,\mu}(\tilde{A}) = P^{t,x,\mu}(A). \quad \square$$

Lemma 6.4. Let $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\mu \in \mathcal{U}^t$. If \mathcal{Y} is an \mathbb{M} -valued, $\mathbf{F}^{P^{t,x,\mu}}$ -adapted process, then $\mathcal{Y}(\mathcal{X}^{t,x,\mu})$ is $\overline{\mathbf{F}}^t$ -adapted. Moreover, if $\mathcal{Y} \in \mathbb{C}_{\mathbf{F}^{P^{t,x,\mu}}}^p([t, T], \mathbb{E}, P^{t,x,\mu})$ (resp. $\mathbb{K}_{\mathbf{F}^{P^{t,x,\mu}}}^p([t, T], P^{t,x,\mu})$) for some $p \in [1, \infty)$, then $\mathcal{Y}(\mathcal{X}^{t,x,\mu}) \in \mathbb{C}_{\overline{\mathbf{F}}^t}^p([t, T], \mathbb{E})$ (resp. $\mathbb{K}_{\overline{\mathbf{F}}^t}^p([t, T])$).

Proof: The first conclusion directly follows from Lemma 5.4. If $\mathcal{Y} \in \mathbb{C}_{\mathbf{F}^{P^{t,x,\mu}}}^p([t, T], \mathbb{E}, P^{t,x,\mu})$ (resp. $\mathbb{K}_{\mathbf{F}^{P^{t,x,\mu}}}^p([t, T], P^{t,x,\mu})$) for some $p \in [1, \infty)$, let $A \triangleq \{\omega \in \Omega^t : \text{the path } s \rightarrow \mathcal{Y}_s(\omega) \text{ is continuous (resp. increasing)}\}$. Then we see from (5.13) that

$$1 = P^{t,x,\mu}(A) = P_0^t \circ (\mathcal{X}^{t,x,\mu})^{-1}(A) = P_0^t(\{\omega \in \Omega^t : \mathcal{X}^{t,x,\mu}(\omega) \in A\}). \quad (6.139)$$

Namely, $\mathcal{Y}(\mathcal{X}^{t,x,\mu})$ has P_0^t -a.s. continuous (resp. increasing) paths. Applying (5.13) again yields that

$$E_t \left[\sup_{s \in [t, T]} |\mathcal{Y}_s(\mathcal{X}^{t,x,\mu})|^p \right] = E_{P^{t,x,\mu}} \left[\sup_{s \in [t, T]} |\mathcal{Y}_s|^p \right] < \infty \quad \left(\text{resp. } E_t \left[|\mathcal{Y}_T(\mathcal{X}^{t,x,\mu})|^p \right] = E_{P^{t,x,\mu}} \left[|\mathcal{Y}_T|^p \right] < \infty \right).$$

Thus, $\mathcal{Y}(\mathcal{X}^{t,x,\mu}) \in \mathbb{C}_{\mathbf{F}^t}^p([t, T], \mathbb{E})$ (resp. $\mathbb{K}_{\mathbf{F}^t}^p([t, T])$). \square

Lemma 6.5. *Let $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\mu \in \mathcal{U}^t$. If \mathcal{Z} is an \mathbb{M} -valued, $\mathbf{F}^{P^{t,x,\mu}}$ -progressively measurable process, then $\mathcal{Z}(\mathcal{X}^{t,x,\mu})$ is $\bar{\mathbf{F}}^t$ -progressively measurable. Consequently, for any $p \in [1, \infty)$ if $\mathcal{Z} \in \mathbb{H}_{\mathbf{F}^{P^{t,x,\mu}}}^{p,loc}([t, T], \mathbb{E}, P^{t,x,\mu})$ (resp. $\mathbb{H}_{\mathbf{F}^{P^{t,x,\mu}}}^{p,\hat{p}}([t, T], \mathbb{E}, P^{t,x,\mu})$ for some $\hat{p} \in [1, \infty)$), then $\mathcal{Z}(\mathcal{X}^{t,x,\mu}) \in \mathbb{H}_{\bar{\mathbf{F}}^t}^{p,loc}([t, T], \mathbb{E})$ (resp. $\mathbb{H}_{\bar{\mathbf{F}}^t}^{p,\hat{p}}([t, T], \mathbb{E})$).*

Proof: Let \mathcal{Z} be an \mathbb{M} -valued, $\mathbf{F}^{P^{t,x,\mu}}$ -progressively measurable process. Given $s \in [t, T]$, we define $\Pi_{t,s}^{x,\mu}(r, \omega) \triangleq (r, \mathcal{X}^{t,x,\mu}(\omega))$, $\forall (r, \omega) \in [t, s] \times \Omega^t$. For any $\mathcal{E} \in \mathcal{B}([t, s])$ and $A \in \mathcal{F}_s^{P^{t,x,\mu}}$, Lemma 5.4 implies that

$$(\Pi_{t,s}^{x,\mu})^{-1}(\mathcal{E} \times A) = \{(r, \omega) \in [t, s] \times \Omega^t : (r, \mathcal{X}^{t,x,\mu}(\omega)) \in \mathcal{E} \times A\} = \mathcal{E} \times (\mathcal{X}^{t,x,\mu})^{-1}(A) \in \mathcal{B}([t, s]) \otimes \bar{\mathcal{F}}_s^t.$$

Hence, the rectangular measurable set $\mathcal{E} \times A \in \Lambda_s \triangleq \{\mathcal{D} \subset [t, s] \times \Omega^t : (\Pi_{t,s}^{x,\mu})^{-1}(\mathcal{D}) \in \mathcal{B}([t, s]) \otimes \bar{\mathcal{F}}_s^t\}$, which is clearly a σ -field of $[t, s] \times \Omega^t$. It follows that $\mathcal{B}([t, s]) \otimes \mathcal{F}_s^{P^{t,x,\mu}} \subset \Lambda_s$. For any $\mathcal{M} \in \mathcal{B}(\mathbb{M})$, since $\mathcal{Z}^{-1}(\mathcal{M}) \triangleq \{(r, \omega) \in [t, s] \times \Omega^t : \mathcal{Z}_r(\omega) \in \mathcal{M}\} \in \mathcal{B}([t, s]) \otimes \mathcal{F}_s^{P^{t,x,\mu}} \subset \Lambda_s$,

$$\begin{aligned} \{(r, \omega) \in [t, s] \times \Omega^t : \mathcal{Z}_r(\mathcal{X}^{t,x,\mu}(\omega)) \in \mathcal{M}\} &= \{(r, \omega) \in [t, s] \times \Omega^t : (r, \mathcal{X}^{t,x,\mu}(\omega)) \in \mathcal{Z}^{-1}(\mathcal{M})\} \\ &= (\Pi_{t,s}^{x,\mu})^{-1}(\mathcal{Z}^{-1}(\mathcal{M})) \in \mathcal{B}([t, s]) \otimes \bar{\mathcal{F}}_s^t. \end{aligned}$$

Hence, $\mathcal{Z}(\mathcal{X}^{t,x,\mu})$ is $\bar{\mathbf{F}}^t$ -progressively measurable.

If $\mathcal{Z} \in \mathbb{H}_{\mathbf{F}^{P^{t,x,\mu}}}^{p,loc}([t, T], \mathbb{E}, P^{t,x,\mu})$ for some $p \in [1, \infty)$, let $A \triangleq \{\omega \in \Omega^t : \int_t^T |\mathcal{Z}_s(\omega)|^p ds < \infty\}$. Similar to (6.139), we see from (5.13) that $1 = P^{t,x,\mu}(A) = P_0^t(\{\omega \in \Omega^t : \mathcal{X}^{t,x,\mu}(\omega) \in A\})$, i.e., $\mathcal{Z}(\mathcal{X}^{t,x,\mu})$ has P_0^t -a.s. p -integrable paths. Thus $\mathcal{Z}(\mathcal{X}^{t,x,\mu}) \in \mathbb{H}_{\bar{\mathbf{F}}^t}^{p,loc}([t, T], \mathbb{E})$.

Moreover, if $\mathcal{Z} \in \mathbb{H}_{\mathbf{F}^{P^{t,x,\mu}}}^{p,\hat{p}}([t, T], \mathbb{E}, P^{t,x,\mu})$ for some $p, \hat{p} \in [1, \infty)$, we can deduce from (5.13) that

$$E_t \left[\left(\int_t^T |\mathcal{Z}_s(\mathcal{X}^{t,x,\mu})|^p ds \right)^{\hat{p}/p} \right] = E_{P^{t,x,\mu}} \left[\left(\int_t^T |\mathcal{Z}_s|^p ds \right)^{\hat{p}/p} \right] < \infty.$$

Therefore, $\mathcal{Z}(\mathcal{X}^{t,x,\mu}) \in \mathbb{H}_{\bar{\mathbf{F}}^t}^{p,\hat{p}}([t, T], \mathbb{E})$. \square

Lemma 6.6. *Let $(t, x) \in [0, T] \times \mathbb{R}^d$. For any $P \in \mathcal{Q}_S^{t,x}$, the \mathbf{F}^t -adapted continuous process $M_s^t \triangleq B_s^t - \int_t^s b(r, B_r^t) dr$, $s \in [t, T]$ is a continuous martingale with respect to (\mathbf{F}^t, P) . Consequently,*

$$W_s^P = \int_{[t,s]}^P (\hat{g}_r^t)^{-1} dM_r^t, \quad s \in [t, T]. \quad (6.140)$$

Proof: We fix $\mu \in \mathcal{U}^t$ and let $\tilde{\mathcal{N}}_\mu \triangleq \{X_r^{t,x,\mu} \neq \tilde{X}_r^{t,x,\mu}, \exists r \in [t, T]\} \in \mathcal{N}^{P_0^t}$. Given $t \leq s < r \leq T$, since $\int_t^s \mu_r dB_r^t$, $s \in [t, T]$ is a martingale with respect to $(\bar{\mathbf{F}}^t, P_0^t)$, for any finite subset $\{t_1 < \dots < t_m\}$ of $\mathbb{Q} \cap [t, s]$ and any

$\{(x_i, \lambda_i)\}_{i=1}^m \subset \mathbb{Q}^d \times \mathbb{Q}_+$, we can deduce from (5.7) and the $\bar{\mathbf{F}}^t$ -adaptedness of $X^{t,x,\mu}$ that

$$\begin{aligned} & \int_{\omega' \in \bigcap_{i=1}^m (B_{t_i}^t)^{-1}(O_{\lambda_i}(x_i))} (M_r^t(\omega') - M_s^t(\omega')) dP^{t,x,\mu}(\omega') \\ &= \int_{\omega \in (\mathcal{X}^{t,x,\mu})^{-1}\left(\bigcap_{i=1}^m (B_{t_i}^t)^{-1}(O_{\lambda_i}(x_i))\right)} (B_r^t(\mathcal{X}^{t,x,\mu}(\omega)) - B_s^t(\mathcal{X}^{t,x,\mu}(\omega)) - \int_s^r b(r', (x + B_{r'}^t(\mathcal{X}^{t,x,\mu}(\omega)))) dr') dP_0^t(\omega) \\ &= \int_{\omega \in (\mathcal{N}_\mu^{t,x} \cup \tilde{\mathcal{N}}_\mu)^c \cap (\mathcal{X}^{t,x,\mu})^{-1}\left(\bigcap_{i=1}^m (B_{t_i}^t)^{-1}(O_{\lambda_i}(x_i))\right)} (X_r^{t,x,\mu}(\omega) - X_s^{t,x,\mu}(\omega) - \int_s^r b(r', X_{r'}^{t,x,\mu}(\omega)) dr') dP_0^t(\omega) \\ &= \int_{\omega \in \bigcap_{i=1}^m (X_{t_i}^{t,x,\mu})^{-1}(O_{\lambda_i}(x_i))} \left(\int_s^r \mu_{r'}(\omega) dB_{r'}^t(\omega)\right) dP_0^t(\omega) = 0. \end{aligned}$$

This shows $\mathcal{C}_s^t \subset \Lambda_s^t \triangleq \{A \in \mathcal{F}_s^t : \int_A (M_r^t - M_s^t) dP^{t,x,\mu} = 0\}$. As Λ_s^t is clearly a Dynkin system, we see from Lemma 1.1 and Dynkin system theorem that $\mathcal{F}_s^t = \sigma(\mathcal{C}_s^t) \subset \Lambda_s^t$, which implies that $E_{P^{t,x,\mu}}[M_r^t | \mathcal{F}_s^t] = M_s^t$, $P^{t,x,\mu}$ -a.s. Hence, M^t is a continuous martingale with respect to $(\mathbf{F}^t, P^{t,x,\mu})$. By Lemma 1.3 (1), M^t is also a continuous martingale with respect to $(\mathbf{F}^{P^{t,x,\mu}}, P^{t,x,\mu})$. As

$$\mathfrak{J}_s^{P^{t,x,\mu}} = \int_{[t,s]}^{P^{t,x,\mu}} (\hat{q}_r^t)^{-1} dB_r^t = \int_t^s (\hat{q}_r^t)^{-1} b(r, x + B_r^t) dr + \int_{[t,s]}^{P^{t,x,\mu}} (\hat{q}_r^t)^{-1} dM_r^t, \quad s \in [t, T],$$

we see that the process $\left\{ \int_{[t,s]}^{P^{t,x,\mu}} (\hat{q}_r^t)^{-1} dM_r^t \right\}_{s \in [t, T]}$ is exactly the martingale part $W^{P^{t,x,\mu}}$ of $\mathfrak{J}^{P^{t,x,\mu}}$. \square

Proof of Proposition 5.3: Fix $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\mu \in \mathcal{U}^t$. As $M_s^t = B_s^t - \int_t^s b(r, x + B_r^t) dr$, $s \in [t, T]$ is a continuous martingale with respect to (\mathbf{F}^t, P) by Lemma 6.6, we see that B^t is a continuous semi-martingale with respect to $(\mathbf{F}^t, P^{t,x,\mu})$. Thus, $P^{t,x,\mu} \in \mathcal{Q}^t$.

Given $i, j \in \{1, \dots, d\}$, it follows from (5.13) that

$$\begin{aligned} 0 &= P^{t,x,\mu} - \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} \left| \langle B^{t,i}, B^{t,j} \rangle_s^{P^{t,x,\mu}} - \sum_{k=1}^{\lfloor 2^n s \rfloor} \left(B_{\frac{k}{2^n}}^{t,i} - B_{\frac{k-1}{2^n}}^{t,i} \right) \left(B_{\frac{k}{2^n}}^{t,j} - B_{\frac{k-1}{2^n}}^{t,j} \right) \right| \\ &= P_0^t - \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} \left| \langle B^{t,i}, B^{t,j} \rangle_s^{P^{t,x,\mu}} (\mathcal{X}^{t,x,\mu}) - \sum_{k=1}^{\lfloor 2^n s \rfloor} \left(\mathcal{X}_{\frac{k}{2^n}}^{t,x,\mu,i} - \mathcal{X}_{\frac{k-1}{2^n}}^{t,x,\mu,i} \right) \left(\mathcal{X}_{\frac{k}{2^n}}^{t,x,\mu,j} - \mathcal{X}_{\frac{k-1}{2^n}}^{t,x,\mu,j} \right) \right| \\ &= P_0^t - \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} \left| \langle B^{t,i}, B^{t,j} \rangle_s^{P^{t,x,\mu}} (\mathcal{X}^{t,x,\mu}) - \sum_{k=1}^{\lfloor 2^n s \rfloor} \left(X_{\frac{k}{2^n}}^{t,x,\mu,i} - X_{\frac{k-1}{2^n}}^{t,x,\mu,i} \right) \left(X_{\frac{k}{2^n}}^{t,x,\mu,j} - X_{\frac{k-1}{2^n}}^{t,x,\mu,j} \right) \right|, \end{aligned}$$

which implies that P_0^t -a.s.,

$$\langle B^{t,i}, B^{t,j} \rangle_s^{P^{t,x,\mu}} (\mathcal{X}^{t,x,\mu}) = \langle X^{t,x,\mu,i}, X^{t,x,\mu,j} \rangle_s^t = \int_t^s \sum_{\ell=1}^d \mu_r^{i\ell} \mu_r^{j\ell} dr, \quad \forall s \in [t, T]. \quad (6.141)$$

Since $\mu_r \in \mathbb{U}_0 = \mathbb{S}_d^{>0}$, $dr \times dP_0^t$ -a.s. (so is $\mu_r^2 = \mu_r \cdot \mu_r$), we can deduce from (6.141), (5.13) and (5.4) that

$$\begin{aligned} 1 &= P_0^t \left\{ \omega \in \Omega^t : s \rightarrow \langle B^t, B^t \rangle_s^{P^{t,x,\mu}} (\mathcal{X}^{t,x,\mu}(\omega)) \text{ is absolutely continuous and} \right. \\ & \quad \left. \lim_{m \rightarrow \infty} m \left(\langle B^t \rangle_s^{P^{t,x,\mu}} - \langle B^t \rangle_{(s-1/m)^+}^{P^{t,x,\mu}} \right) (\mathcal{X}^{t,x,\mu}(\omega)) = \mu_s^2(\omega) \in \mathbb{S}_d^{>0} \text{ for a.e. } s \in [t, T] \right\} \quad (6.142) \end{aligned}$$

$$\begin{aligned} &= P^{t,x,\mu} \left\{ \omega' \in \Omega^t : s \rightarrow \langle B^t, B^t \rangle_s^{P^{t,x,\mu}} (\omega') \text{ is absolutely continuous and} \right. \\ & \quad \left. \hat{a}_s^t(\omega') = \lim_{m \rightarrow \infty} m \left(\langle B^t \rangle_s^{P^{t,x,\mu}} - \langle B^t \rangle_{(s-1/m)^+}^{P^{t,x,\mu}} \right) (\omega') \in \mathbb{S}_d^{>0} \text{ for a.e. } s \in [t, T] \right\}. \quad (6.143) \end{aligned}$$

Therefore, $P^{t,x,\mu} \in \mathcal{Q}_W^t$. \square

Lemma 6.7. Let $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\mu \in \mathcal{U}^t$. It holds P_0^t -a.s. that

$$\hat{q}_s^t(\mathcal{X}^{t,x,\mu}) = \mu_s \text{ for a.e. } s \in [t, T]. \quad (6.144)$$

And for any $\mathcal{Z} \in \mathbb{H}_{\mathbf{F}^t, x, \mu}^{p, loc}([t, T], \mathbb{R}^d, P^{t,x,\mu})$ with $p \in [1, \infty)$, it holds P_0^t -a.s. that

$$\left(\int_{[t,s]}^{P^{t,x,\mu}} \mathcal{Z}_r dW_r^{P^{t,x,\mu}} \right) (\mathcal{X}^{t,x,\mu}) = \int_t^s \mathcal{Z}_r (\mathcal{X}^{t,x,\mu}) dB_r^t, \quad \forall s \in [t, T], \quad (6.145)$$

where $\mathcal{Z}(\mathcal{X}^{t,x,\mu}) \in \mathbb{H}_{\mathbf{F}^t}^{p, loc}([t, T], \mathbb{R}^d)$ by Lemma 6.5.

Proof: We can deduce from Lemma 5.3, (6.143) and (5.13) that

$$\begin{aligned} 1 &= P^{t,x,\mu} \left\{ \omega' \in \Omega^t : (\hat{q}_s^t)^2(\omega') = \hat{a}_s^t(\omega') = \lim_{m \rightarrow \infty} m \left(\langle B^t \rangle_s^{P^{t,x,\mu}} - \langle B^t \rangle_{(s-1/m)^+}^{P^{t,x,\mu}} \right) (\omega') \text{ for a.e. } s \in [t, T] \right\} \\ &= P_0^t \left\{ \omega \in \Omega^t : (\hat{q}_s^t)^2(\mathcal{X}^{t,x,\mu}(\omega)) = \lim_{m \rightarrow \infty} m \left(\langle B^t \rangle_s^{P^{t,x,\mu}} - \langle B^t \rangle_{(s-1/m)^+}^{P^{t,x,\mu}} \right) (\mathcal{X}^{t,x,\mu}(\omega)) \text{ for a.e. } s \in [t, T] \right\}. \end{aligned}$$

This together with (6.142) yields that

$$1 = P_0^t \left\{ \omega \in \Omega^t : (\hat{q}_s^t)^2(\mathcal{X}^{t,x,\mu}(\omega)) = \mu_s^2(\omega) \text{ and } \mu_s(\omega) \in \mathbb{S}_d^{>0} \text{ for a.e. } s \in [t, T] \right\}, \quad (6.146)$$

where we used the fact that $\mu_s \in \mathbb{U}_0$, $ds \times dP_0^t$ -a.s. Since for each $\Gamma \in \mathbb{S}_d^{>0}$ there exists a unique element in $\mathbb{S}_d^{>0}$ such that $\zeta^2 = \varsigma \cdot \varsigma = \Gamma$ (see e.g. Theorem VI.9 of [38]), (6.146) leads to (6.144).

Now, let $\mathcal{Z} \in \mathbb{H}_{\mathbf{F}^t, x, \mu}^{p, loc}([t, T], \mathbb{R}^d, P^{t,x,\mu})$ for some $p \in [1, \infty)$. We have seen from the proof of Proposition 5.3 that the process $M_s^t = B_s^t - \int_t^s b(r, x + B_r^t) dr$, $s \in [t, T]$ is a continuous martingale with respect to $(\mathbf{F}^t, P^{t,x,\mu})$. It is thus a continuous martingale with respect to $(\mathbf{F}^{P^{t,x,\mu}}, P^{t,x,\mu})$ by Lemma 1.3 (1). Then we know (see e.g. Problem 3.2.27 of [27]) that there exists a sequence of \mathbb{R}^d -valued, $\mathbf{F}^{P^{t,x,\mu}}$ -simple processes $\left\{ \Phi_s^n = \sum_{i=1}^{\ell_n} \xi_i^n \mathbf{1}_{\{s \in (t_i^n, t_{i+1}^n]\}} \right\}$, $s \in [t, T]$ (where $t = t_1^n < \dots < t_{\ell_n+1}^n = T$ and $\xi_i^n \in \mathcal{F}_{t_i^n}^{P^{t,x,\mu}}$ for $i = 1, \dots, \ell_n$) such that

$$P^{t,x,\mu} - \lim_{n \rightarrow \infty} \int_t^T \left((\Phi_s^n)^T - (\hat{q}_s^t)^{-1} \mathcal{Z}_s^T \right) d\langle M^t \rangle_s^{P^{t,x,\mu}} \left(\Phi_s^n - \mathcal{Z}_s (\hat{q}_s^t)^{-1} \right) = 0, \quad (6.147)$$

$$\text{and} \quad P^{t,x,\mu} - \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} \left| \sum_{i=1}^{\ell_n} \xi_i^n (M_{s \wedge t_{i+1}^n}^t - M_{s \wedge t_i^n}^t) - \int_{[t,s]}^{P^{t,x,\mu}} \mathcal{Z}_r (\hat{q}_r^t)^{-1} dM_r^t \right| = 0. \quad (6.148)$$

Define a martingale with respect to $(\overline{\mathbf{F}}^t, P_0^t)$: $\Upsilon_s^t \triangleq \int_t^s \mu_r dB_r^t$, $s \in [t, T]$. We can deduce from (6.147), (5.6), (5.13) and (6.144) that

$$\begin{aligned} 0 &= P^{t,x,\mu} - \lim_{n \rightarrow \infty} \int_t^T \left((\Phi_s^n)^T - (\hat{q}_s^t)^{-1} \mathcal{Z}_s^T \right) d\langle B^t \rangle_s^{P^{t,x,\mu}} \left(\Phi_s^n - \mathcal{Z}_s (\hat{q}_s^t)^{-1} \right) \\ &= P_0^t - \lim_{n \rightarrow \infty} \int_t^T \left((\Phi_s^n(\mathcal{X}^{t,x,\mu}))^T - ((\hat{q}_s^t)^{-1} \mathcal{Z}_s^T)(\mathcal{X}^{t,x,\mu}) \right) \hat{a}_s^t(\mathcal{X}^{t,x,\mu}) \left(\Phi_s^n(\mathcal{X}^{t,x,\mu}) - (\mathcal{Z}_s (\hat{q}_s^t)^{-1})(\mathcal{X}^{t,x,\mu}) \right) ds \\ &= P_0^t - \lim_{n \rightarrow \infty} \int_t^T \left((\Phi_s^n(\mathcal{X}^{t,x,\mu}))^T - \mu_s^{-1} \mathcal{Z}_s^T(\mathcal{X}^{t,x,\mu}) \right) d\langle \Upsilon^t \rangle_s^{P_0^t} \left(\Phi_s^n(\mathcal{X}^{t,x,\mu}) - \mathcal{Z}_s(\mathcal{X}^{t,x,\mu}) \mu_s^{-1} \right). \end{aligned} \quad (6.149)$$

Also, (6.148), (6.140) and (5.13) yield that

$$\begin{aligned} 0 &= P^{t,x,\mu} - \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} \left| \sum_{i=1}^{\ell_n} \xi_i^n (M_{s \wedge t_{i+1}^n}^t - M_{s \wedge t_i^n}^t) - \int_{[t,s]}^{P^{t,x,\mu}} \mathcal{Z}_r dW_r^{P^{t,x,\mu}} \right| \\ &= P_0^t - \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} \left| \sum_{i=1}^{\ell_n} \xi_i^n (\mathcal{X}^{t,x,\mu}) \left(\mathcal{X}_{s \wedge t_{i+1}^n}^{t,x,\mu} - \mathcal{X}_{s \wedge t_i^n}^{t,x,\mu} - \int_{s \wedge t_i^n}^{s \wedge t_{i+1}^n} b(r, x + \mathcal{X}_r^{t,x,\mu}) dr \right) - \left(\int_{[t,s]}^{P^{t,x,\mu}} \mathcal{Z}_r dW_r^{P^{t,x,\mu}} \right) (\mathcal{X}^{t,x,\mu}) \right| \\ &= P_0^t - \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} \left| \sum_{i=1}^{\ell_n} \xi_i^n (\mathcal{X}^{t,x,\mu}) \left(\mathcal{X}_{s \wedge t_{i+1}^n}^{t,x,\mu} - \mathcal{X}_{s \wedge t_i^n}^{t,x,\mu} - \int_{s \wedge t_i^n}^{s \wedge t_{i+1}^n} b(r, \mathcal{X}_r^{t,x,\mu}) dr \right) - \left(\int_{[t,s]}^{P^{t,x,\mu}} \mathcal{Z}_r dW_r^{P^{t,x,\mu}} \right) (\mathcal{X}^{t,x,\mu}) \right| \\ &= P_0^t - \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} \left| \sum_{i=1}^{\ell_n} \xi_i^n (\mathcal{X}^{t,x,\mu}) (\Upsilon_{s \wedge t_{i+1}^n}^t - \Upsilon_{s \wedge t_i^n}^t) - \left(\int_{[t,s]}^{P^{t,x,\mu}} \mathcal{Z}_r dW_r^{P^{t,x,\mu}} \right) (\mathcal{X}^{t,x,\mu}) \right|. \end{aligned} \quad (6.150)$$

For $i = 1, \dots, \ell_n$, since $\xi_i^n \in \mathcal{F}_{t_i^n}^{P^{t,x,\mu}}$, we see from Lemma 5.4 that $\xi_i^n(\mathcal{X}^{t,x,\mu}) \in \overline{\mathcal{F}}_{t_i^n}^t$. Thus, $\Phi^n(\mathcal{X}^{t,x,\mu})$ is also an $\overline{\mathbf{F}}^t$ -simple process. Proposition 3.2.26 of [27] and (6.149) then imply that

$$P_0^t - \lim_{n \rightarrow \infty} \sup_{s \in [t, T]} \left| \sum_{i=1}^{\ell_n} \xi_i^n(\mathcal{X}^{t,x,\mu}) (X_{s \wedge t_{i+1}^n}^{t,x,\mu} - X_{s \wedge t_i^n}^{t,x,\mu}) - \int_t^s \mathcal{Z}_r(\mathcal{X}^{t,x,\mu}) \mu_r^{-1} d\Upsilon_r^t \right| = 0,$$

which together with (6.150) shows that P_0^t -a.s.

$$\left(\int_{[t,s]} \mathcal{Z}_r dW_r^{P^{t,x,\mu}} \right) (\mathcal{X}^{t,x,\mu}) = \int_t^s \mathcal{Z}_r(\mathcal{X}^{t,x,\mu}) \mu_r^{-1} d\Upsilon_r^t = \int_t^s \mathcal{Z}_r(\mathcal{X}^{t,x,\mu}) dB_r^t, \quad \forall s \in [t, T]. \quad \square$$

Proof of Proposition 5.4: Let $(t, x) \in [0, T] \times \mathbb{R}^d$ and $\mu \in \mathcal{U}^t$. For any $s \in [t, T]$, since $W_s^{P^{t,x,\mu}} \in \mathcal{F}_s^{P^{t,x,\mu}}$, one can easily deduce that

$$\mathcal{G}_s^{P^{t,x,\mu}} = \sigma\left(\sigma(W_r^{P^{t,x,\mu}}, r \in [t, s]) \cup \mathcal{N}^{P^{t,x,\mu}}\right) \subset \mathcal{F}_s^{P^{t,x,\mu}}.$$

As to the inverse inclusion, (6.145) implies that $B_s^t = W_s^{P^{t,x,\mu}}(\mathcal{X}^{t,x,\mu})$ holds except on a P_0^t -null set \mathcal{N}_0 . Given $r \in [t, s]$ and $\mathcal{E} \in \mathcal{B}(\mathbb{R}^d)$, one has

$$(B_r^t)^{-1}(\mathcal{E}) = \{\omega \in \Omega^t : B_r^t(\omega) \in \mathcal{E}\} = \{\omega \in \Omega^t : W_r^{P^{t,x,\mu}}(\mathcal{X}^{t,x,\mu}(\omega)) \in \mathcal{E}\} \Delta \mathcal{N}_{\mathcal{E}}$$

with $\mathcal{N}_{\mathcal{E}} \triangleq \{\omega \in \Omega^t : B_r^t(\omega) \in \mathcal{E}\} \Delta \{\omega \in \Omega^t : W_r^{P^{t,x,\mu}}(\mathcal{X}^{t,x,\mu}(\omega)) \in \mathcal{E}\} \subset \mathcal{N}_0$. To wit,

$$(B_r^t)^{-1}(\mathcal{E}) \in \Lambda_s^t \triangleq \left\{ A \subset \Omega^t : A = \left((\mathcal{X}^{t,x,\mu})^{-1}(\tilde{A}) \right) \Delta \mathcal{N} \text{ for some } \tilde{A} \in \mathcal{G}_s^{P^{t,x,\mu}} \text{ and } \mathcal{N} \in \mathcal{N}^{P_0^t} \right\}.$$

Clearly, $\left\{ (\mathcal{X}^{t,x,\mu})^{-1}(\tilde{A}) : \tilde{A} \in \mathcal{G}_s^{P^{t,x,\mu}} \right\}$ is a σ -field of Ω^t . Similar to Problem 2.7.3 of [27], one can show that Λ_s^t forms a σ -field of Ω^t . It follows that $\overline{\mathcal{F}}_s^t \subset \Lambda_s^t$. As $\mathcal{N}^{P_0^t} \subset \Lambda_s^t$, we further have $\overline{\mathcal{F}}_s^t \subset \Lambda_s^t$.

Now, for any $A \in \mathcal{F}_s^{P^{t,x,\mu}}$, we know from Lemma 5.4 that $(\mathcal{X}^{t,x,\mu})^{-1}(A) \in \overline{\mathcal{F}}_s^t \subset \Lambda_s^t$. Hence, there exists $\tilde{A} \in \mathcal{G}_s^{P^{t,x,\mu}} \subset \mathcal{F}_s^{P^{t,x,\mu}}$ and $\mathcal{N} \in \mathcal{N}^{P_0^t}$ such that $(\mathcal{X}^{t,x,\mu})^{-1}(A) = (\mathcal{X}^{t,x,\mu})^{-1}(\tilde{A}) \Delta \mathcal{N}$, which leads to that

$$\mathcal{N} = (\mathcal{X}^{t,x,\mu})^{-1}(A) \Delta (\mathcal{X}^{t,x,\mu})^{-1}(\tilde{A}) = (\mathcal{X}^{t,x,\mu})^{-1}(A \Delta \tilde{A}).$$

Then (5.13) shows that $P^{t,x,\mu}(A \Delta \tilde{A}) = P_0^t(\mathcal{N}) = 0$, namely, $A \Delta \tilde{A} \in \mathcal{N}^{P^{t,x,\mu}}$. It follows that $A = \tilde{A} \Delta (A \Delta \tilde{A}) \in \mathcal{G}_s^{P^{t,x,\mu}}$, thus $\mathcal{F}_s^{P^{t,x,\mu}} = \mathcal{G}_s^{P^{t,x,\mu}}$. \square

Proof of Proposition 5.5: Let us simply denote $(\mathcal{Y}^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x})), \mathcal{Z}^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x})), \underline{\mathcal{K}}^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x})), \overline{\mathcal{K}}^{t,x,P^{t,x,\mu}}(T, h(B_T^{t,x})))$ by $(\mathcal{Y}, \mathcal{Z}, \underline{\mathcal{K}}, \overline{\mathcal{K}})$. Lemma 6.4 and Lemma 6.5 shows that

$$(\mathcal{Y}, \mathcal{Z}, \underline{\mathcal{K}}, \overline{\mathcal{K}}) \triangleq (\mathcal{Y}(\mathcal{X}^{t,x,\mu}), \mathcal{Z}(\mathcal{X}^{t,x,\mu}), \underline{\mathcal{K}}(\mathcal{X}^{t,x,\mu}), \overline{\mathcal{K}}(\mathcal{X}^{t,x,\mu})) \in \mathbb{G}_{\overline{\mathbf{F}}^t}^q([t, T]).$$

And we can deduce from (5.13), Lemma 6.7 that $(\mathcal{Y}, \mathcal{Z}, \underline{\mathcal{K}}, \overline{\mathcal{K}})$ satisfies

$$\begin{cases} \mathcal{Y}_s = h(\tilde{X}_s^{t,x,\mu}) + \int_s^T f(r, \tilde{X}_r^{t,x,\mu}, \mathcal{Y}_r, \mathcal{Z}_r, \mu_r) dr + \underline{\mathcal{K}}_T - \underline{\mathcal{K}}_s - (\overline{\mathcal{K}}_T - \overline{\mathcal{K}}_s) - \int_s^T \mathcal{Z}_r dB_r^t, & s \in [t, T], \\ \underline{L}_s^{t,x,\mu} \leq \mathcal{Y}_s \leq \overline{L}_s^{t,x,\mu}, & s \in [t, T], \quad \text{and} \quad \int_t^T (\mathcal{Y}_s - \underline{L}_s^{t,x,\mu}) d\underline{\mathcal{K}}_s = \int_t^T (\overline{L}_s^{t,x,\mu} - \mathcal{Y}_s) d\overline{\mathcal{K}}_s = 0 \end{cases}$$

on the probability space $(\Omega^t, \overline{\mathcal{F}}_T^t, P_0^t)$. Since this doubly Reflected BSDE admits a unique solution in $\mathbb{G}_{\overline{\mathbf{F}}^t}^q([t, T])$ according to Section 2, we have

$$(\mathcal{Y}, \mathcal{Z}, \underline{\mathcal{K}}, \overline{\mathcal{K}}) = (Y^{t,x,\mu}(T, h(\tilde{X}_T^{t,x,\mu})), Z^{t,x,\mu}(T, h(\tilde{X}_T^{t,x,\mu})), \underline{K}^{t,x,\mu}(T, h(\tilde{X}_T^{t,x,\mu})), \overline{K}^{t,x,\mu}(T, h(\tilde{X}_T^{t,x,\mu}))). \quad \square$$

References

- [1] R. ATAR AND A. BUDHIRAJA, *A stochastic differential game for the inhomogeneous ∞ -Laplace equation*, Ann. Probab., 38 (2010), pp. 498–531.
- [2] E. BAYRAKTAR AND Y. HUANG, *On the multi-dimensional controller and stopper games*, (2011). Available at <http://arxiv.org/abs/1009.0932>.
- [3] E. BAYRAKTAR, I. KARATZAS, AND S. YAO, *Optimal stopping for dynamic convex risk measures*, to appear in the Illinois Journal of Mathematics (special issue on honor of Don. Burkholder), (2012), pp. 1–45. Available at <http://lanl.arxiv.org/abs/0909.4948>.
- [4] E. BAYRAKTAR AND M. SÎRBU, *Stochastic perron’s method and verification without smoothness using viscosity comparison: obstacle problems and Dynkin games*, (2011). Available at <http://arxiv.org/abs/1112.4904>.
- [5] E. BAYRAKTAR AND S. YAO, *(A longer version of) On zero-sum stochastic differential games*, (2011). Available at <http://arxiv.org/abs/1112.5744>.
- [6] E. BAYRAKTAR AND S. YAO, *Optimal stopping for non-linear expectations—Part I*, Stochastic Process. Appl., 121 (2011), pp. 185–211.
- [7] ———, *Optimal stopping for non-linear expectations—Part II*, Stochastic Process. Appl., 121 (2011), pp. 212–264.
- [8] S. BROWNE, *Stochastic differential portfolio games*, J. Appl. Probab., 37 (2000), pp. 126–147.
- [9] R. BUCKDAHN AND J. LI, *Stochastic differential games and viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equations*, SIAM J. Control Optim., 47 (2008), pp. 444–475.
- [10] ———, *Probabilistic interpretation for systems of Isaacs equations with two reflecting barriers*, NoDEA Nonlinear Differential Equations Appl., 16 (2009), pp. 381–420.
- [11] ———, *Stochastic differential games with reflection and related obstacle problems for Isaacs equations*, Acta Math. Appl. Sin. Engl. Ser., 27 (2011), pp. 647–678.
- [12] P. CHERIDITO, H. M. SONER, N. TOUZI, AND N. VICTOIR, *Second-order backward stochastic differential equations and fully nonlinear parabolic PDEs*, Comm. Pure Appl. Math., 60 (2007), pp. 1081–1110.
- [13] J. CVITANIĆ AND I. KARATZAS, *Backward stochastic differential equations with reflection and Dynkin games*, Ann. Probab., 24 (1996), pp. 2024–2056.
- [14] J. DUGUNDJI, *Topology*, Allyn and Bacon Inc., Boston, Mass., 1966.
- [15] B. EL ASRI, S. HAMADÈNE, AND H. WANG, *L^p -solutions for doubly reflected backward stochastic differential equations*, to appear in Stochastic Analysis and Applications, (2011).
- [16] N. EL-KAROUI AND S. HAMADÈNE, *BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations*, Stochastic Process. Appl., 107 (2003), pp. 145–169.
- [17] W. H. FLEMING AND D. HERNÁNDEZ HERNÁNDEZ, *On the value of stochastic differential games*, Communications on Stochastic Analysis, 5 (2011), pp. 341–351.
- [18] W. H. FLEMING AND P. E. SOUGANIDIS, *On the existence of value functions of two-player, zero-sum stochastic differential games*, Indiana Univ. Math. J., 38 (1989), pp. 293–314.
- [19] S. HAMADÈNE AND M. HASSANI, *BSDEs with two reflecting barriers: the general result.*, Probab. Theory Relat. Fields, 132 (2005), pp. 237–264.

- [20] S. HAMADÈNE AND J. P. LEPELTIER, *Backward equations, stochastic control and zero-sum stochastic differential games*, Stochastics Stochastics Rep., 54 (1995), pp. 221–231.
- [21] ———, *Zero-sum stochastic differential games and backward equations*, Systems Control Lett., 24 (1995), pp. 259–263.
- [22] S. HAMADÈNE AND J.-P. LEPELTIER, *Reflected BSDEs and mixed game problem*, Stochastic Process. Appl., 85 (2000), pp. 177–188.
- [23] S. HAMADÈNE, J.-P. LEPELTIER, AND Z. WU, *Infinite horizon reflected backward stochastic differential equations and applications in mixed control and game problems*, Probab. Math. Statist., 19 (1999), pp. 211–234.
- [24] S. HAMADÈNE AND A. POPIER, *L^p -solutions for reflected backward stochastic differential equations*, to appear in Stochastics and Dynamics.
- [25] S. HAMADÈNE, E. ROTENSTEIN, AND A. ZĂLINESCU, *A generalized mixed zero-sum stochastic differential game and double barrier reflected BSDEs with quadratic growth coefficient*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.), 55 (2009), pp. 419–444.
- [26] R. L. KARANDIKAR, *On pathwise stochastic integration*, Stochastic Process. Appl., 57 (1995), pp. 11–18.
- [27] I. KARATZAS AND S. E. SHREVE, *Brownian motion and stochastic calculus*, vol. 113 of Graduate Texts in Mathematics, Springer-Verlag, New York, second ed., 1991.
- [28] I. KARATZAS AND W. D. SUDDERTH, *The controller-and-stopper game for a linear diffusion*, Ann. Probab., 29 (2001), pp. 1111–1127.
- [29] I. KARATZAS AND I.-M. ZAMFIRESCU, *Martingale approach to stochastic differential games of control and stopping*, Ann. Probab., 36 (2008), pp. 1495–1527.
- [30] N. V. KRYLOV, *Controlled diffusion processes*, vol. 14 of Stochastic Modelling and Applied Probability, Springer-Verlag, Berlin, 2009. Translated from the 1977 Russian original by A. B. Aries, Reprint of the 1980 edition.
- [31] A. J. LAZARUS, D. E. LOEB, J. G. PROPP, AND D. ULLMAN, *Richman games*, in Games of no chance (Berkeley, CA, 1994), vol. 29 of Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, Cambridge, 1996, pp. 439–449.
- [32] A. MATOUSSI, D. POSSAMAI, AND C. ZHOU, *Second Order Reflected Backward Stochastic Differential Equations*, 2012. Available at <http://arxiv.org/abs/1201.0746>.
- [33] M. NUTZ, *A quasi-sure approach to the control of non-Markovian stochastic differential equations*, tech. rep., ETH Zurich, 2011. Available at <http://arxiv.org/abs/1106.3273>.
- [34] S. PENG, *G-Brownian motion and dynamic risk measure under volatility uncertainty*, 2007. Available at <http://lanl.arxiv.org/abs/0711.2834>.
- [35] ———, *G-expectation, G-Brownian motion and related stochastic calculus of Itô type*, in Stochastic analysis and applications, vol. 2 of Abel Symp., Springer, Berlin, 2007, pp. 541–567.
- [36] S. PENG AND X. XU, *BSDEs with random default time and related zero-sum stochastic differential games*, C. R. Math. Acad. Sci. Paris, 348 (2010), pp. 193–198.
- [37] Y. PERES, O. SCHRAMM, S. SHEFFIELD, AND D. B. WILSON, *Tug-of-war and the infinity Laplacian*, J. Amer. Math. Soc., 22 (2009), pp. 167–210.
- [38] M. REED AND B. SIMON, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York, 1972.

- [39] D. REVUZ AND M. YOR, *Continuous martingales and Brownian motion*, vol. 293 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, third ed., 1999.
- [40] H. L. ROYDEN, *Real analysis*, Macmillan Publishing Company, New York, third ed., 1988.
- [41] H. SONER, N. TOUZI, AND J. ZHANG, *Dual formulation of second order target problems*, preprint, (2011). Available at <http://arxiv.org/abs/1003.6050>.
- [42] ———, *Martingale representation theorem for the G -expectation*, Stochastic Process. Appl., 121 (2011), pp. 265–287.
- [43] ———, *Quasi-sure stochastic analysis through aggregation*, to appear in Electronic Journal of Probability, (2011). Available at <http://arxiv.org/abs/1003.4431>.
- [44] ———, *Wellposedness of second order Backward SDEs*, to appear in Probability Theory and Related Fields, (2011). Available at <http://arxiv.org/abs/1003.6053>.
- [45] D. W. STROOCK AND S. R. S. VARADHAN, *Multidimensional diffusion processes*, Classics in Mathematics, Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.